

Spring Topology and Dynamical Systems Conference
Special Session on Dynamics

Realizing full n -shifts in simple Smale flows

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Smale Flows

Let \mathcal{M} be a smooth compact connected 3-manifold with or without boundary.

A nonsingular Smale flow (**NSF**) on \mathcal{M} is a structurally stable flow with one-dimensional chain recurrent set \mathcal{R} . It is hyperbolic on \mathcal{R} and the stable and unstable manifolds only meet transversely. The term was introduced by John Franks in the 1970's.

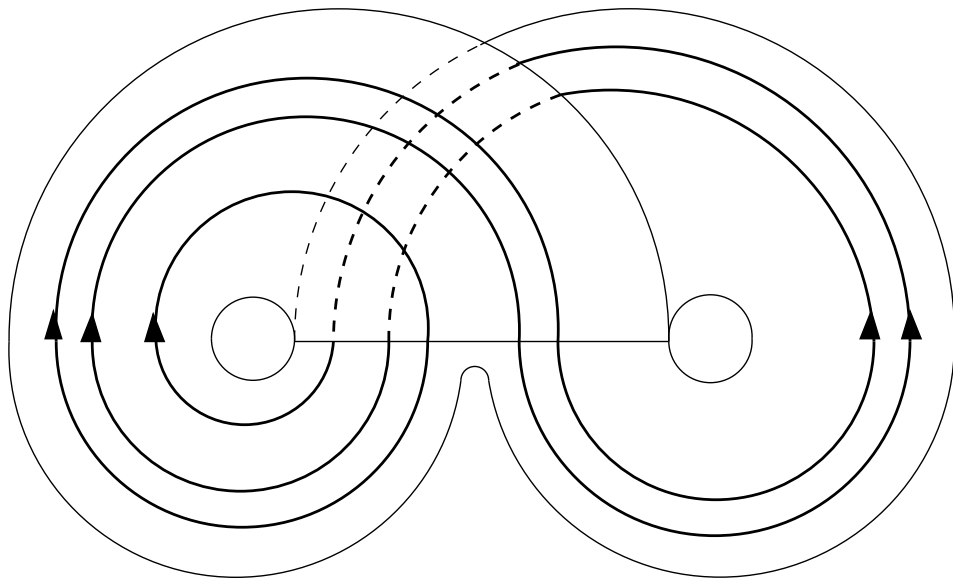
The chain recurrent set, by a theorem of Smale, consists of a finite number of disjoint **basic sets**, which are compact and transitive. A basic set may be an **attractor**, **repeller** or **saddle set**.

Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial shift of finite type - we call these **chaotic saddle sets**.

Knots and Templates

Basic sets that are closed orbits form knots. While chaotic basic sets contain infinitely many knots and knot types. [Franks & Williams]

To get a handle on chaotic saddle sets we use **templates**. These are branched two manifolds with semi-flows formed by taking an isolating neighborhood and collapsing out the local stable manifolds. [Birman & Williams]

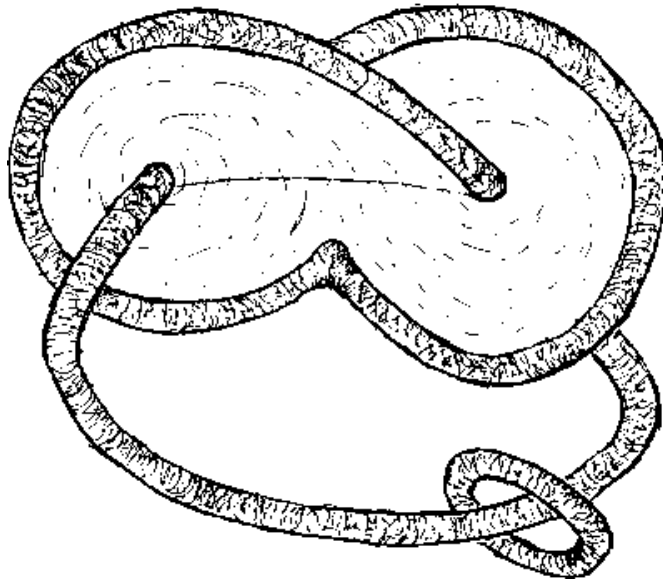


Simple Smale Flows

We will be working with nonsingular Smale flows on S^3 with three basic sets:

- An attracting closed orbit, a
- A repelling closed orbit, r
- A chaotic saddle set, cs

Isolating nbhds of the basic sets can be glued together, using diffeomorphisms that match the vector fields, to form the ambient manifold, S^3



Full N-shift Spaces

DEFINITION: Given a set of n symbols the full n -shift space is the set of all bi-infinite sequences of the symbols (i.e., there are no forbidden words), together with a shift homeomorphism.

Using a theorem of Franks [1985], it is easy to show that each n -shift there is a simple Smale flow whose saddle set has the n -shift map conjugate to the first return map of some cross section.

Franks theorem does not tell us the knot types of a and r .

Main Theorem

THEOREM: For every $n \geq 2$ there exists a simple Smale flow where a and r are unknotted and the saddle set has a cross section whose first return map is conjugate to the full n -shift.

Outline of Proof.

The proof is pictorial and uses induction on n .

The induction step is really a two-step.

If the claim holds for an even value of n , then it holds for $n + 1$.

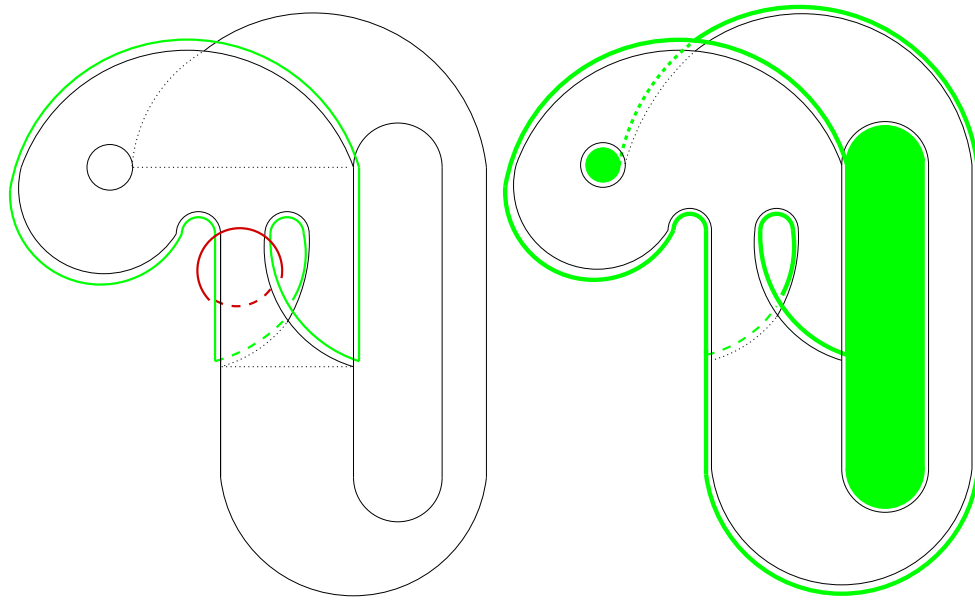
If the claim holds for an odd value of n , then it holds for $n + 1$.

The details of the arguments are different.

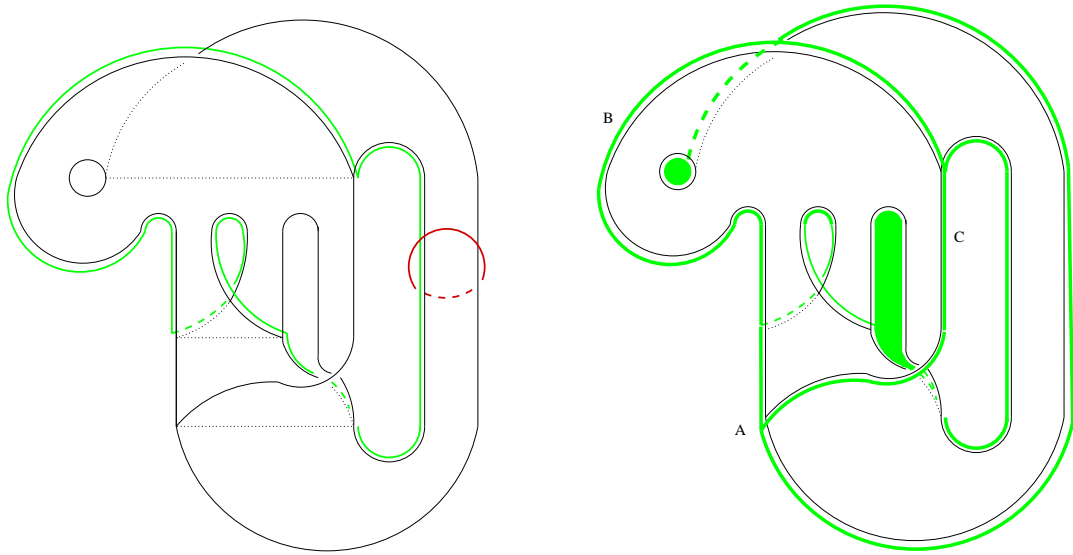
For $n = 2$ the result is known. [S 2000] We will show examples for $n = 3, 4, 5, 6$ and discuss the induction step.

Note: the theorem holds for $n = 1$ if we interpret it as a Morse-Smale flow with one saddle orbit.

N = 3

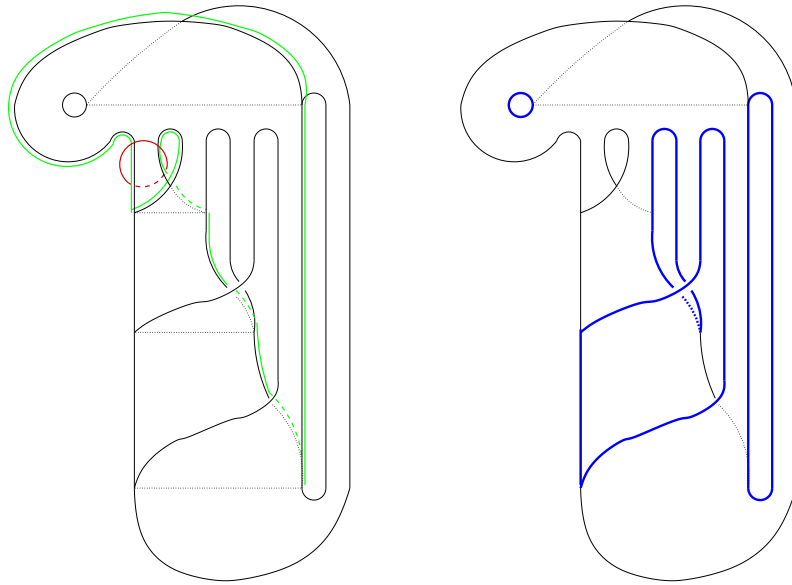


$N=4$

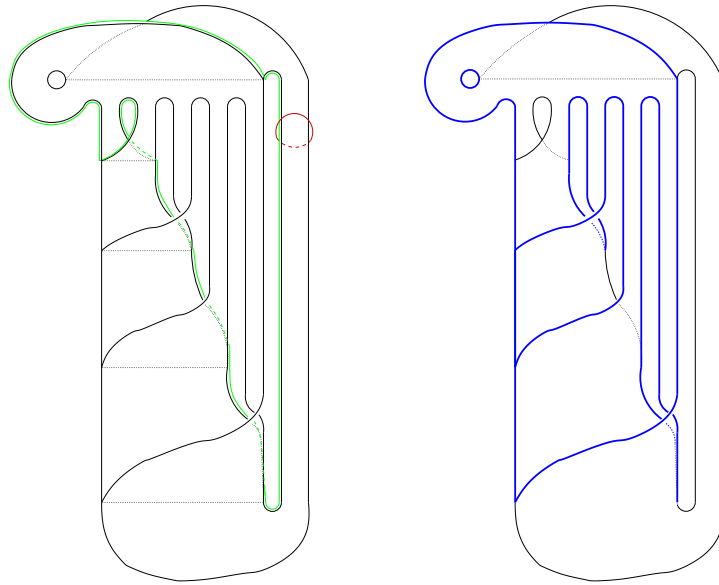


Cap off along loop ABC with $D \times I$.

N=5



N=6



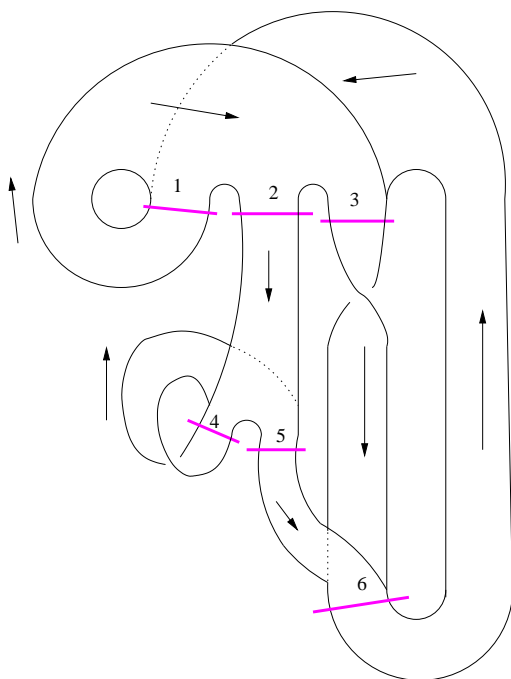
Incidence and Structure Matrices

Let D_1, D_2, \dots, D_n be cross sectional disks for Markov partition. Assume they are “small enough” for the definitions below to work.

Incidence Matrix: Let $A = [a_{ij}]$ be given by $a_{ij} = 1$ if there is a orbit directly from D_i to D_j and be zero otherwise.

Structure Matrix: Let $S = [s_{ij}]$ be given by $s_{ij} = \pm a_{ij}$, with $-$ meaning the first return map is orientation reversing.

Example:



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Linking Numbers

THEOREM (Franks 1981): In a simple Smale flow the unsigned linking number of a and r is the determinant of I minus the structure matrix:

$$|\text{lk}(a, r)| = |\det(I - S)|$$

Corollary: For a simple Smale flow with saddle set a suspension of the n -shift we have

$$\text{lk}(a, r) = \begin{cases} \text{even if } n \text{ is odd} \\ \text{odd if } n \text{ is even} \end{cases}$$

Conjecture

$l \setminus n$	2	3	4	5	6	7	8	...
0	N	Y	N	?	N	?	N	...
1	Y	N	Y	N	Y	N	Y	...
2	N	Y	N	Y	N	Y	N	...
3	?	N	?	N	?	N	?	...
4	N	?	N	?	N	?	N	...
5	?	N	?	N	?	N	?	...
6	N	?	N	?	N	?	N	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Conjecture: All ? = Y.

What about allowed knot types? Whitehead links?

Main References

John Franks

Nonsingular Smale Flows on S^3
Topology, 24 (3) (1985) 265–282

M.S.

Visually building Smale flows in S^3
Topology and Its Applications
106 (2000), no. 1, 1–19.