Realizing full n-shifts in simple Smale flows

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Smale Flows

Let $\mathcal{M}$ be a smooth compact connected 3-manifold with or without boundary.

A nonsingular Smale flow **(NSF)** on $\mathcal{M}$ is a structurally stable flow with one-dimensional chain recurrent set $\mathcal{R}$. It is hyperbolic on $\mathcal{R}$ and the stable and unstable manifolds only meet transversely. The term was introduced by John Franks in the 1970's.

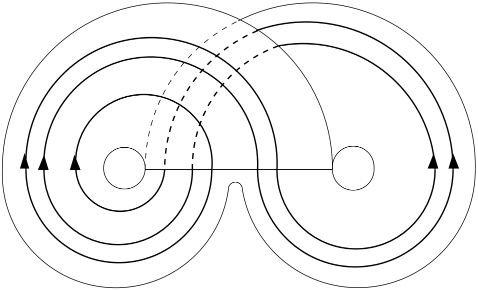
The chain recurrent set, by a theorem of Smale, consists of a finite number of disjoint **basic sets**, which are compact and transitive. A basic set may be an **attractor**, **repeller** or **saddle set**.

Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial shift of finite type - we call these **chaotic saddle sets**.

Knots and Templates

Basic sets that are closed orbits form knots. Chaotic basic sets contain infinitely many knots and knot types. [Franks & Williams, 1985]

To get a handle on chaotic saddle sets we use **templates**. These are branched two manifolds with semi-flows formed by taking an isolating neighborhood and collapsing out the local stable manifolds. [Birman & Williams, 1983]

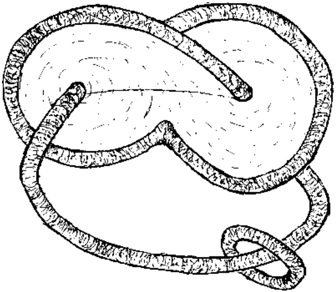


Simple Smale Flows

We will be working with nonsingular Smale flows on S^3 with three basic sets:

- An attracting closed orbit, a
- A repelling closed orbit, r
- A chaotic saddle set, cs

Isolating nbhds of the basic sets can be glued together, using diffeomorphisms that match the vector fields, to form the ambient manifold, S^3.



Full N-shift Spaces

Definition

Given a set of n symbols the full n-shift space is the set of all bi-infinite sequences of the symbols (i.e., there are no forbidden words), together with a shift homoemorphism.

Using a theorem of Franks [1985], it is easy to show that for each n-shift there exists a simple Smale flow whose saddle set has the n-shift map conjugate to the first return map of some cross section.

Franks' theorem does not tell us the knot types of a and r.

Incidence and Structure Matrices

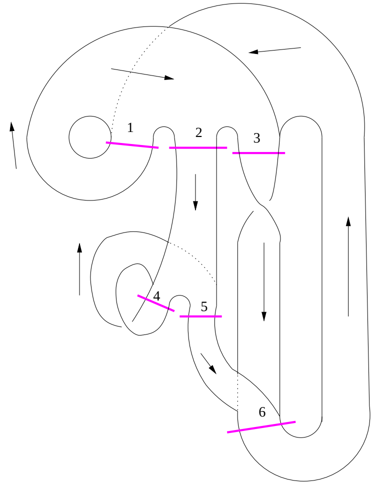
But, another result of Franks' will give the unsigned linking number of a and r.

Let $D_1, D_2, \ldots, D_n$ be cross sectional disks for Markov partition. Assume these are "small enough" for the definitions below to work.

Incidence Matrix: Let $A = [a_{ij}]$ be given by $a_{ij} = 1$ if there is a orbit directly from D_i to D_j and be zero otherwise.

Structure Matrix: Let $S = [s_{ij}]$ be given by $s_{ij} = \pm a_{ij}$, with "$-$" meaning the first return map is orientation reversing.

An Example



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Linking Numbers

Theorem (Franks 1981)

In a simple Smale flow the unsigned linking number of a and r is the determinant of I minus the structure matrix,

$$|lk(a, r)| = |\det(I - S)| .$$

Corollary

For a simple Smale flow with saddle set a suspension of the n-shift we have

$$lk(a, r) = \begin{cases} \text{even if } n \text{ is odd,} \\ \text{odd if } n \text{ is even.} \end{cases}$$

Main Theorem

Theorem

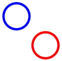
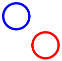
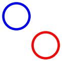
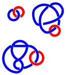
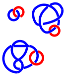
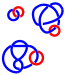
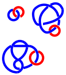
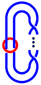
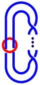
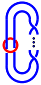
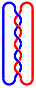
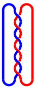
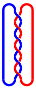
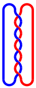
A. *Let $n \geq 3$ be odd. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, with $a \cup r$ unlinked unknots, s.t. (i) the a (r) links every closed saddle orbit but one and the r (a) links no other closed orbits, (ii) both a and r link every closed saddle orbit but one, or (iii) neither links any other closed orbits.*

B. *Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $lk(a, r) = 1$ and the pair $a \cup r$ can be any of, (i) a Hopf link, (ii) a trefoil and meridian, or (iii) a figure-8 knot and meridian.*

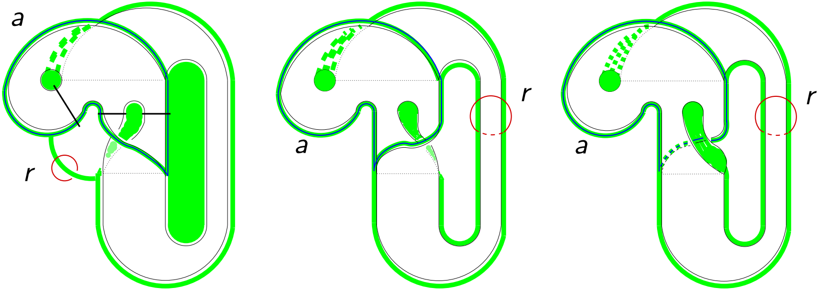
C. *Let $n \geq 3$ be odd and p be any integer. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $lk(a, r) = 2$, and a (resp. r) has braid word σ^{2p+1} and r (resp. a) is an unknot serving as a braid axis.*

D. *Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $lk(a, r) = 3$, and the braid word for $a \cup r$ is σ^6.*

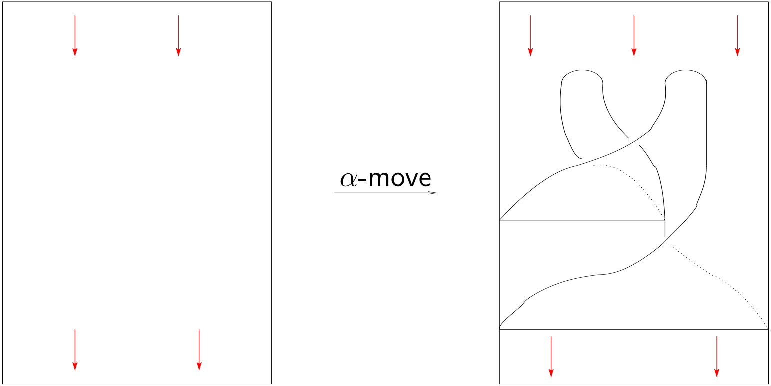
Table

l \ n	2	3	4	5	6	7	8	
0								•••
1								•••
2								•••
3								•••
4		?		?		?		
5	?		?		?		?	

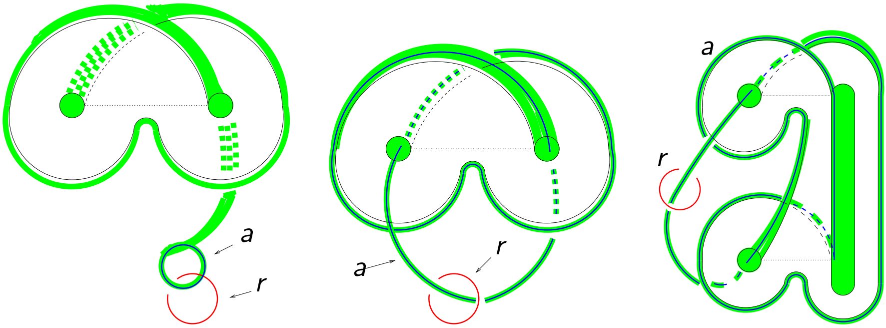
Case A. $N = 3$ $L = 0$



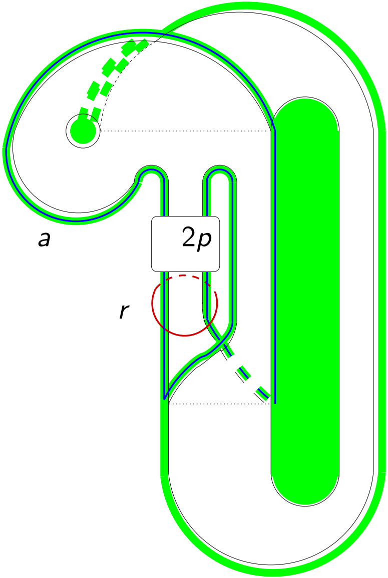
Induction Step



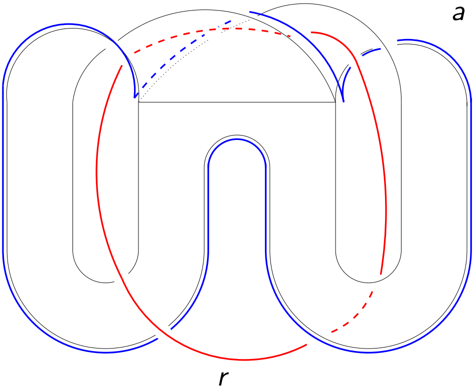
Case B



Case C



Case D



Conjecture

$l \backslash n$	2	3	4	5	6	7	8	...
0	N	Y	N	Y	N	Y	N	...
1	Y	N	Y	N	Y	N	Y	...
2	N	Y	N	Y	N	Y	N	...
3	Y	N	Y	N	Y	N	Y	...
4	N	?	N	?	N	?	N	...
5	?	N	?	N	?	N	?	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Conjecture

All ? = Y.

Questions

What about allowed knot types? Whitehead links?

Remark

A not as yet submitted paper by Beguin, Bonatti and Bin Yu gives some restrictions on the link type of $a \cup r$.

For example you cannot have two unlinked nontrivial knots

~~Their work seems to exclude the Whitehead link although they do not state this.~~

References

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