Periodic orbits in a chaotic attractor introduced by Clark Robinson

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Lorenz Attractor & Templates

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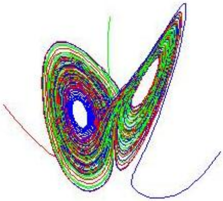

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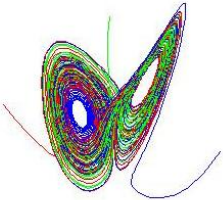

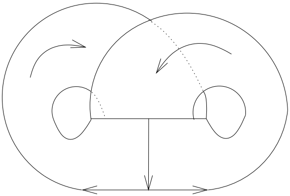

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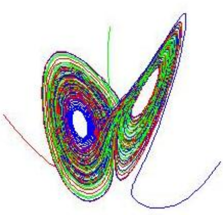

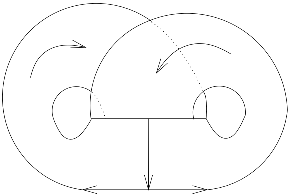

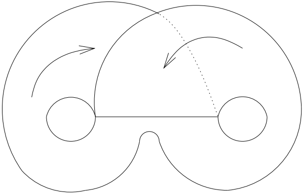

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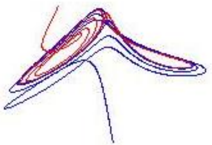

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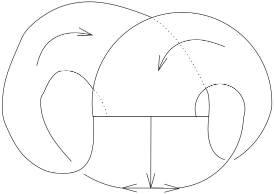

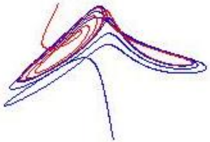

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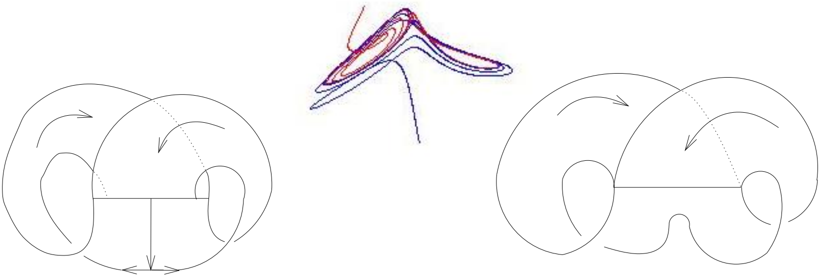

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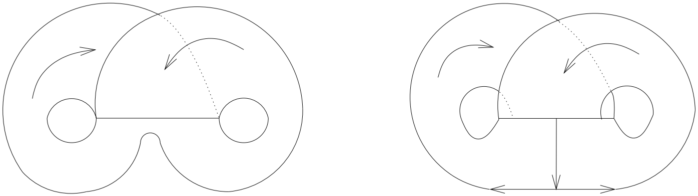

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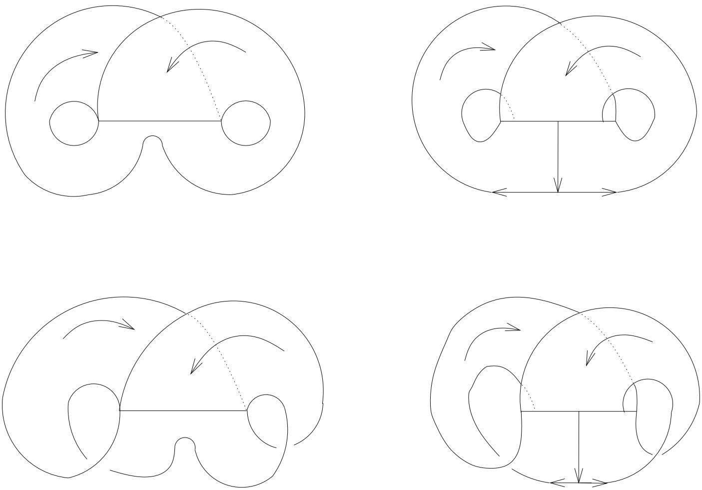

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On $L(-1,-1)$ the situation is a bit different. For orbits in $L(-1,-1)$ we have the following.

a. The orbit for xy is unlinked with all other closed orbits.

b. The orbit for x is unlinked to orbits of the form xy^n and the orbit for y is unlinked to orbits of the form x^ny,

c. Any pair of closed orbits not covered by (a) or (b) are linked.

For the proof see the next frame.

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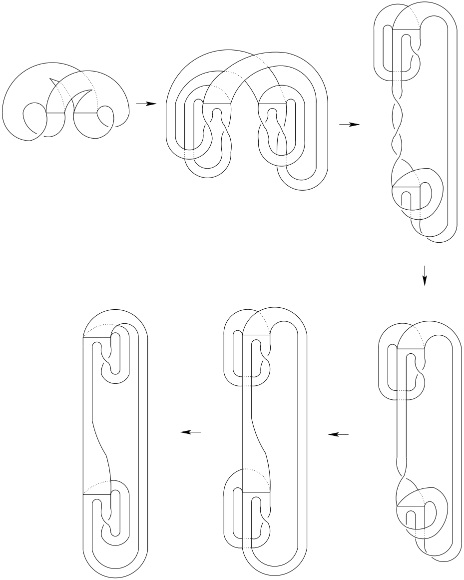

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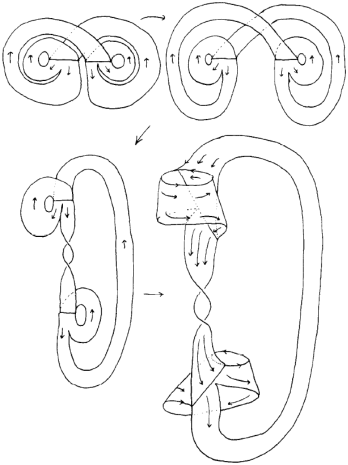

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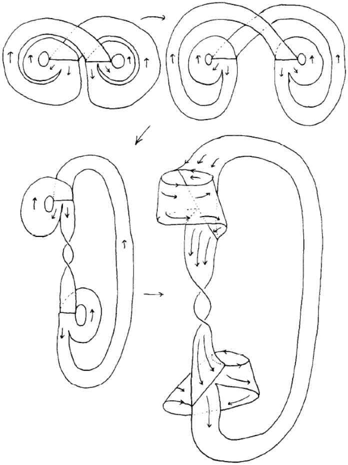

Figure is from Birman and Williams' 1983 paper

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Fact: Positive braids are fibered. [Stallings]

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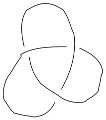

Left–hand Trefoil
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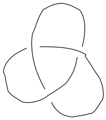

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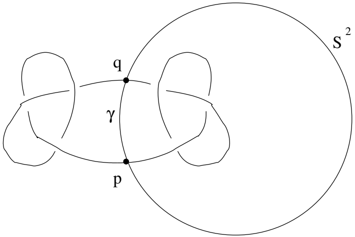
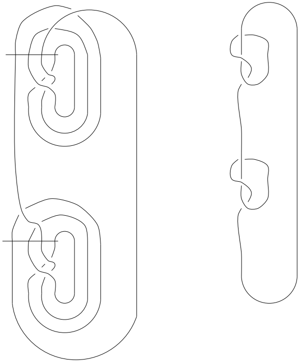


Square Knot
Composite

$L(-1,-1)$ has compisite knots



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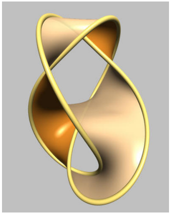

Figure from: http://www.win.tue.nl/ ∼vanwijk/seifertview/knot_gallery.htm

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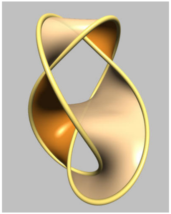

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If $S^3 - N(K)$ can be fibered by a Seifert surface of K, then K is a fibered knot.

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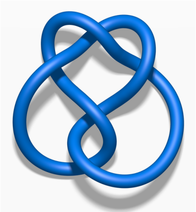

http://katlas.org/wiki/File:Blue_Three-Twist_Knot.png

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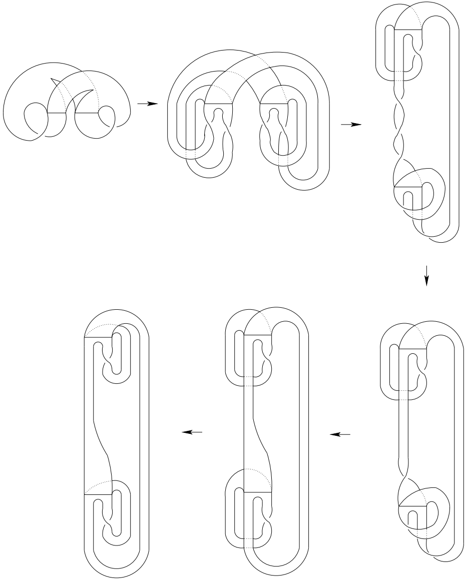

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Example: The orbit with word $xy^4x^2yx^4y^2$ in $L(-1,-1)$ can be presented as the following braid on five strands, $(32233232221\bar{4})^2$. A calculation shows that its Conway polynomial has leading coefficient 3. [S., 2005] Hence it is not a positive braid. [James M. van Buskirk, 1983]

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The proof is a bit complicated to present here, but it closely follows Stallings' proof that postive briads are fibered. We give an brief outline.

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If ν_* is an isomorphism, then K is fibered. [Stallings]

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Then we show that having all the twists of the same type forces ν_* to be surjective.

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Thus, while it is known that any template supports infinitely many distinct knot types the collection of prime knots in $L(-1,-1)$ seems rather narrow.