Realizing full n-shifts in simple Smale flows

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Flows

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We will mostly work with the 3-sphere.

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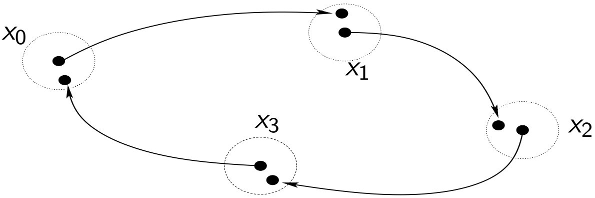

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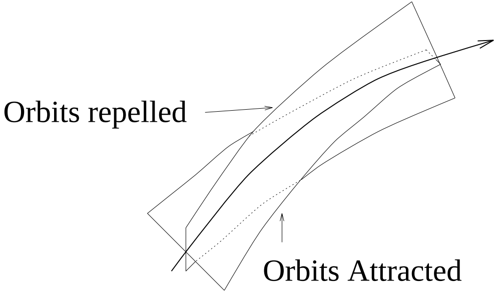

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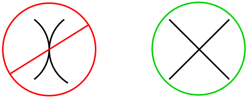

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The term was introduced by John Franks in the 1970's.

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Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial **shift of finite type** - we call these **chaotic saddle sets**.

Shifts of Finite Type

An example: Let Σ be all bi-infinite sequences of A's and B's for which AA never appears. Thus,

$$...BBBABABA.BABABBABABB... \in \Sigma.$$

Let $\sigma : \Sigma \to \Sigma$ be the shift map; it shifts the "decimal point" one place to the right.

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With an obvious metric this Σ is a Cantor space and σ is a homeomorphism.

SFT via a Graph or Matrix

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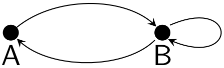

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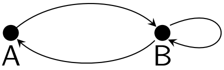

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

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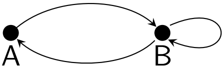

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Note: If the matrix is irreducible and not a permutation matrix, the SFT will be a Cantor space.

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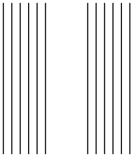
Take $\Sigma \times [0, 1]$,



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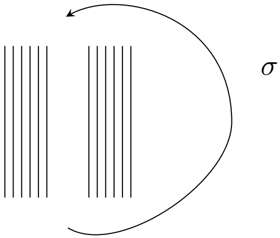
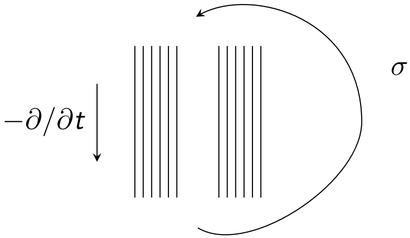
Take $\Sigma \times [0,1]$, glue $\Sigma \times 0$ to $\Sigma \times 1$ via σ,



Suspensions of SFTs

Take $\Sigma \times [0,1]$, glue $\Sigma \times 0$ to $\Sigma \times 1$ via σ, and add the vector field $-\partial/\partial t$.



Knots and Templates

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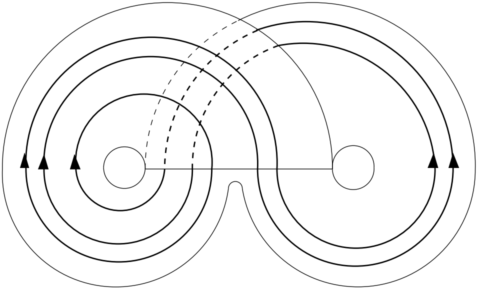

To get a handle on chaotic saddle sets we use **templates**. These are branched two manifolds with semi-flows formed by taking an isolating neighborhood and collapsing out the local stable manifolds. [Birman & Williams, 1983]

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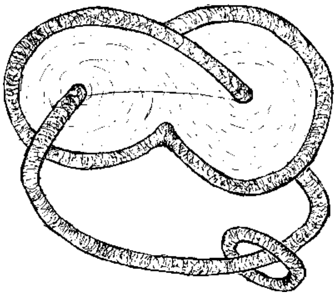
2006 - Ghys established a connection with the periodic orbits in the geodesic flow on a modular surface.
2009 - Birman & Kofman characterize Lorenz links as iteratively twisted torus links and that both hyperbolic volume and Mahler measure of Jones polynomials are bounded for generic infinite families of Lorenz links. The precise characterization of those Lorenz links that are hyperbolic is the main open question. (From the MR review.)

Simple Smale Flows

We will be working with nonsingular Smale flows on S^3 with three basic sets:

- An attracting closed orbit, a
- A repelling closed orbit, r
- A chaotic saddle set, cs

Isolating nbhds of the basic sets can be glued together, using diffeomorphisms that match the vector fields, to form the ambient manifold, S^3.



Lorenz-Smale Flows

Definition

A *Lorenz-Smale flow* is a simple Smale where the saddle set's template is an embedding of the Lorenz template.

Theorem (S. 2000)

For a Lorenz-Smale flow on S^3, the link $a \cup r$ is either a Hopf link or a trefoil and meridian.

Lorenz-like Smale flows

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Definition

We use $L(0,0)$ to denote the Lorenz template, $L(0,1)$ to denote a variation where one band contains a Möbius band, and $L(1,1)$ to denote a variation where each band contains a Möbuis band.

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Theorem (Bin Yu, 2009)

For an L(1, 0) Lorenz-like Smale flow on S^3 *the* $a \cup r$ *link is either a Hopf link or a trefoil and meridian pair.*

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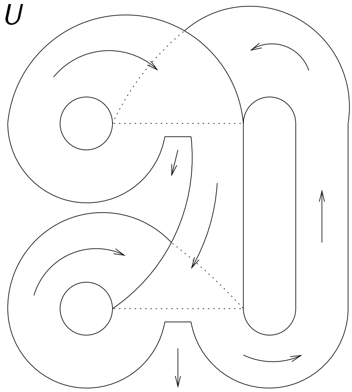
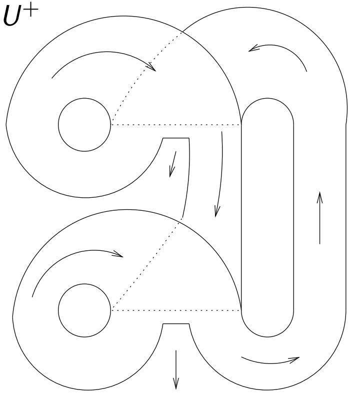
Bin Yu goes on to describe all 3-manifolds that can support these cases.

Four-band Templates

The template U contains all knots and links [Ghrist 1997] while U^+ has only prime positive braids [S. 1994]. Yet they are flow equivalent.





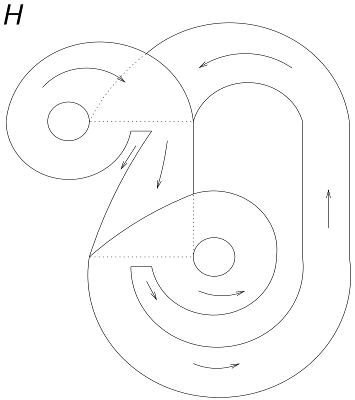
Four-band Templates

Theorem (Haynes & S. 2014)

A simple Smale flow with saddle set modeled by U will have $a \cup r$ a Hopf link or a figure-8 knot and meridian; if the saddle set is modeled by U^+ then $a \cup r$ a Hopf link or a trefoil and meridian.

Four-band Templates

The template H contains all knots and links [Ghrist 1997 + S. 1994] while H^+ has only positive braids with at most two prime factors [S. 1994]. Yet they are flow equivalent.



Four-band Templates

Theorem (Adhikari 2015)

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Full N-shift Spaces

Definition

A SFT with an alphabet of n symbols and no forbidden words is called **the full n-shift space**.

Using a theorem of Franks [1985], it is easy to show that for each n-shift there exists a simple Smale flow whose saddle set has the n-shift map conjugate to the first return map of some cross section.

Franks' theorem does not tell us the knot types of a and r.

Incidence and Structure Matrices

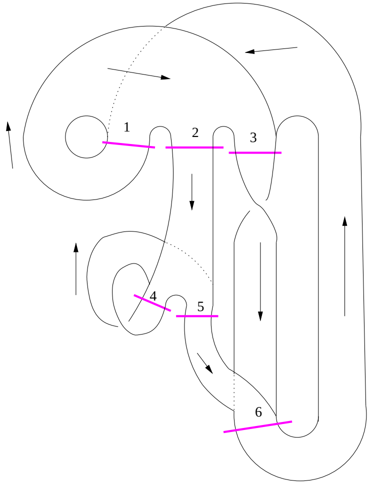
But, another result of Franks' will give the unsigned linking number of a and r.

Let $D_1, D_2, \ldots, D_n$ be cross sectional disks for Markov partition of the saddle set. Assume these are "small enough" for the definitions below to work.

Incidence Matrix: Let $A = [a_{ij}]$ be given by $a_{ij} = 1$ if there is a orbit directly from D_i to D_j and be zero otherwise.

Structure Matrix: Let $S = [s_{ij}]$ be given by $s_{ij} = \pm a_{ij}$, with "$-$" meaning the first return map is orientation reversing.

An Example



Incidence Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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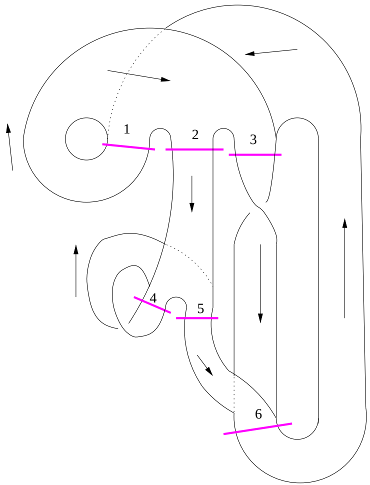

Structure Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Linking Numbers

Theorem (Franks 1981)

In a simple Smale flow the unsigned linking number of a and r is the determinant of I minus the structure matrix,

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Corollary

For a simple Smale flow with saddle set a suspension of the n-shift we have

$$lk(a, r) = \begin{cases} \text{even if } n \text{ is odd,} \\ \text{odd if } n \text{ is even.} \end{cases}$$

Main Theorem

A. Let $n \geq 3$ be odd. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, with $a \cup r$ unlinked unknots, s.t. (i) the a (r) links every closed saddle orbit but one and the r (a) links no other closed orbits, (ii) both a and r link every closed saddle orbit but one, or (iii) neither links any other closed orbits.

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B. Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $\text{lk}(a, r) = 1$ and the pair $a \cup r$ can be any of, (i) a Hopf link, (ii) a trefoil and meridian, or (iii) a figure-8 knot and meridian.

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C. Let $n \geq 3$ be odd and p be any integer. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $\text{lk}(a, r) = 2$, and a (resp. r) has braid word σ^{2p+1} and r (resp. a) is an unknot serving as a braid axis.

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D. Let $n \geq 2$ be even. There exists a simple Smale flow on S^3 such that the saddle set is a suspension of a full n-shift, $\text{lk}(a, r) = 3$, and the braid word for $a \cup r$ is σ^6.

Table

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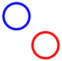
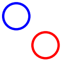
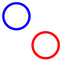
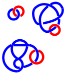
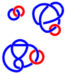
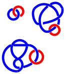
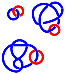
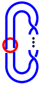
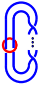
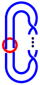
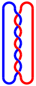
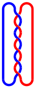
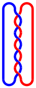
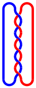
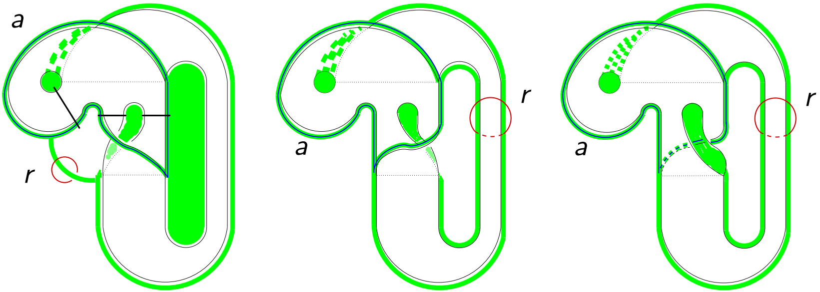
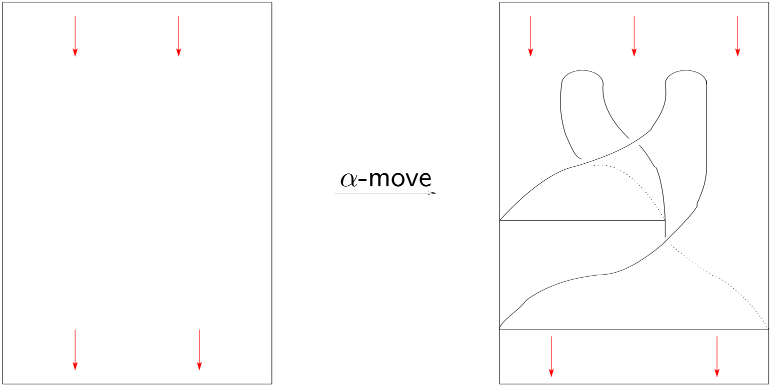
0								•••
1								•••
2								•••
3								•••
4		?		?		?		
5	?		?		?		?	

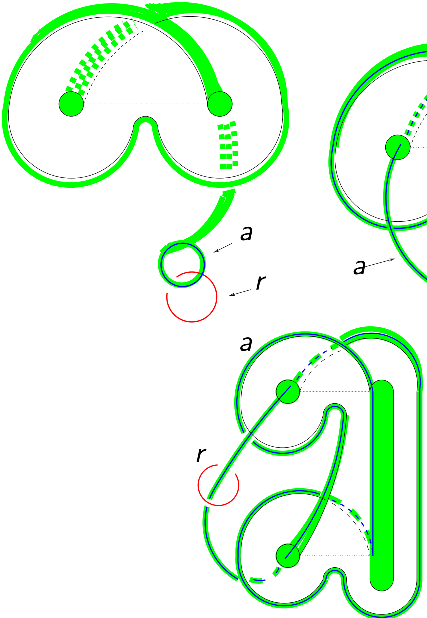
Case A. $N = 3$ $L = 0$



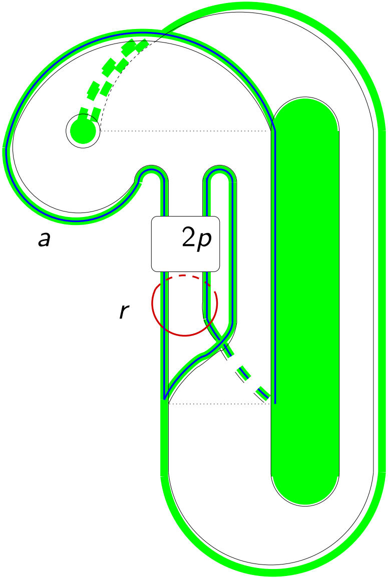
Induction Step



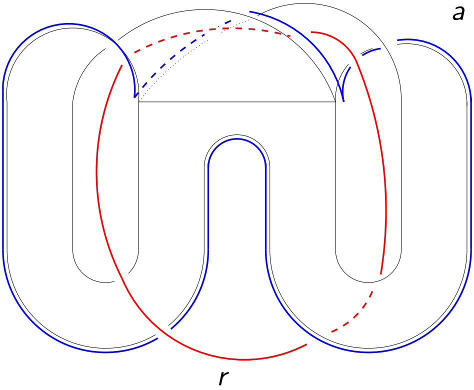
Case B



Case C



Case D



Conjecture

$l \backslash n$	2	3	4	5	6	7	8	$\dots$
0	N	Y	N	Y	N	Y	N	$\dots$
1	Y	N	Y	N	Y	N	Y	$\dots$
2	N	Y	N	Y	N	Y	N	$\dots$
3	Y	N	Y	N	Y	N	Y	$\dots$
4	N	?	N	?	N	?	N	$\dots$
5	?	N	?	N	?	N	?	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Conjecture

All ? = Y.

Questions

What about allowed knot types? Whitehead links?

Remark

A not as yet submitted paper by Beguin, Bonatti and Bin Yu gives some restrictions on the link type of $a \cup r$ for any simple Smale flow.

For example you cannot have two unlinked nontrivial knots.

Some References

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