Transverse Foliations to nonsingular Morse-Smale flows and Bott-integrable Hamiltonian systems

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Nonsingular Morse-Smale Flows

DEFINITION. A flow ϕ on a manifold M is a **Morse-Smale flow** if the following hold.

- The chain recurrent set is hyperbolic. Each component is called a **Basic set**.
- The stable and unstable manifolds of basic sets meet transversely.
- Each basic set consists of a single closed orbit or fixed point.

For M a compact manifold, it follows that Morse-Smale flows have a finite number of periodic orbits and fixed points. A **nonsingular flow** is a flow without fixed points.

Notation: The basic sets are indexed by: 0 for an attractor, 1 for a saddle, and 2 for a repeller.

Not all 3-manifolds can support NMS flows. John Morgan has given a theorem that characterizes just which 3-manifolds do. (In higher dimensions Asimov has shown that all manifolds with Euler characteristic 0 support NMS flows.)

Wada's Theorem

Which indexed links can be realized as invariant sets of NMS flows of S^3?

WADA'S THEOREM. Let $\mathcal{F}$ be the set of indexed links which can be realized as the collection of periodic orbits of a nonsingular Morse-Smale flow on S^3, respecting index. Then $\mathcal{F} = \mathcal{W}$, where $\mathcal{W}$ is defined on the next few transparencies.

Wada's Links

Let $\mathcal{W}$ be the collection of indexed links determined by the following axioms:

W0: The Hopf link indexed by 0 and 2 is in $\mathcal{W}$.

W1: If $L_1, L_2 \in \mathcal{W}$ then $L_1 \circ L_2 \circ u \in \mathcal{W}$, where u (here and below) is an unknot in S^3 indexed by 1.

W2: If $L_1, L_2 \in \mathcal{W}$ and K_2 is a component of L_2 indexed by 0 or 2, then $L_1 \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W3: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 with indices 0 and 2 (resp.), then $(L_1 - K_1) \circ (L_2 - K_2) \circ u \in \mathcal{W}$.

W4: If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 (resp.) each with index 0 or 2, then

$$((L_1, K_1)\#(L_2, K_2)) \cup m \in \mathcal{W},$$

where $K_1\#K_2$ shares the index of either K_1 or K_2 and m is a meridian of $K_1\#K_2$ indexed by 1.

W5: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L replacing a tubular neighborhood of K with a solid torus with three closed orbits, K_1, K_2, and K_3. K_1 is the core and so has the same knot type as K. K_2 and K_3 are parallel (p, q) cables of K_1. The index of K_2 is 1. The indices of K_1 and K_3 may be either 0 or 2, but at least one of them must be equal to the index of K.

W6: If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L by changing the index of K to 1 and placing a $(2, q)$-cable of K in a tubular neighborhood of K, indexed by i.

W7: $\mathcal{W}$ is minimal. That is, $\mathcal{W} \subset \mathcal{W}'$ for any collection, $\mathcal{W}'$, satisfying W0-W6.

Remark 1 The last condition, W7, means that $\mathcal{W}$ is generated from the indexed Hopf link in S^3 by applying operations W1-W6.

Bolt-Integrable Hamiltonian Systems

In 1998 Casasayas, Alfaro, & Nunes studied indexed links of fixed points of flows induced by certain Hamiltonian systems.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6.

Contact Flows: Ghrist & Etnyre

In 1999 Ghrist & Etnyre studied gradient flows of 3-manifolds tangent to plane fields associated to a contact structure. In these flows there are indexed links of fixed points.

They showed that these links were a subset of the NMS links.

On S^3 they showed that this subset of $\mathcal{W}$ is generated by W0, W4, W5, & W6.

Transverse Foliations to Flows

Definition 1 A **2-dimensional foliation** $\mathcal{F} = \{L_\alpha\}$ of a 3-manifold M is a partition of M such that $\forall x \in M$ $\exists$ a chart $(U_x, \phi : U_x \to \mathbb{R}^3)$ such each connected component of $\phi(U \cap L_\alpha)$ is of the form $\{(x, y, z) \in \phi(U_x) \,|\, z = \text{ a constant}\}$. The L_α's are called the leaves of the foliation.

Definition 2 An indexed link on a 3-manifold has the **Linking Property** if for every closed orbit that bounds a disk, there is an attracting or repelling closed orbit that has nonzero algebraic linking number with that disk.

Theorem 1 (Goodman; see also Yano) A nonsingular Morse-Smale flow on a 3-manifold has a transverse 2-dimensional foliation (each flow line meets any leaf transversely) if and only if its periodic orbits satisfy the linking property.

Wada moves and Transverse Foliations

Theorem 2 (S) The set of indexed links that can be realized as the set of periodic orbits of nonsingular Morse-Smale flows on S^3 that have transverse foliations is the subset of $\mathcal{W}$ generated by W0, W4, W5, & W6.

Future work: Lens Spaces

Can one generalize Wada's Theorem to Lens Spaces?

For $S^2 \times S^1$ this is very hard since one can have transverse surfaces that are not tori: spheres, bi-tori and Klein bottles can be realized.

None-the-less a series of papers by Cordero, Martinez Alfaro, & Vindel, based on Cordero's dissertation give a partial answer. They have a sequence of Wada-like moves that generate NMS flows on $S^2 \times S^1$. They haven't succeeded in showing these moves are complete.

A the subset of their moves that generate indexed links associated to Hamiltonian systems correspond to NMS flows with transverse foliations. But, again completeness is unknown.

For "true" Lens spaces transverse spheres and bi-tori do not arise but sometimes Klein bottles can.

PROPOSITION (Bredon & Woods, 1969): The Lens space $L(p, q)$ contains an embedded Klein bottle if and only if $p = 4k$ and $q = 2k - 1$ for some $k \geq 1$.