# Realizing full n-shifts in simple Smale flows


Michael Sullivan
mikesullivan (at) math (dot) siu (dot) edu
http://www.math.siu.edu/sullivan

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# Smale Flows

Let $\mathcal{M}$ be a smooth compact connected 3-manifold with or without boundary.

A nonsingular Smale flow **(NSF)** on $\mathcal{M}$ is a structurally stable flow with one-dimensional chain recurrent set $\mathcal{R}$. It is hyperbolic on $\mathcal{R}$ and the stable and unstable manifolds only meet transversely. The term was introduced by John Franks in the 1970's.

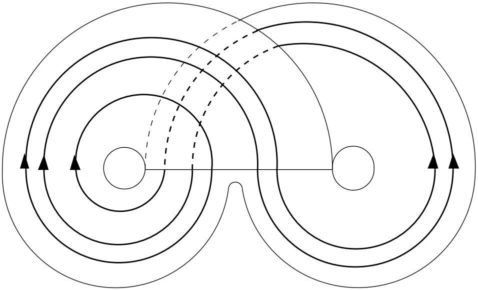
The chain recurrent set, by a theorem of Smale, consists of a finite number of disjoint **basic sets**, which are compact and transitive. A basic set may be an **attractor**, **repeller** or **saddle set**.

Attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a nontrivial shift of finite type - we call these **chaotic saddle sets**.

# Knots and Templates

Basic sets that are closed orbits form knots. Chaotic basic sets contain infinitely many knots and knot types. [Franks & Williams, 1985]

To get a handle on chaotic saddle sets we use **templates**. These are branched two manifolds with semi-flows formed by taking an isolating neighborhood and collapsing out the local stable manifolds. [Birman & Williams, 1983]

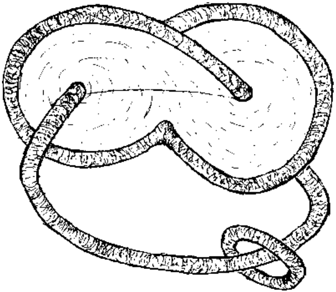


# Simple Smale Flows

We will be working with nonsingular Smale flows on $S^3$ with three basic sets:

- An attracting closed orbit, $a$
- A repelling closed orbit, $r$
- A chaotic saddle set, $cs$

Isolating nbhds of the basic sets can be glued together, using diffeomorphisms that match the vector fields, to form the ambient manifold, $S^3$.



# Full N-shift Spaces

**Definition**

Given a set of $n$ symbols the full $n$-shift space is the set of all bi-infinite sequences of the symbols (i.e., there are no forbidden words), together with a shift homoemorphism.

Using a theorem of Franks [1985], it is easy to show that for each $n$-shift there exists a simple Smale flow whose saddle set has the $n$-shift map conjugate to the first return map of some cross section.

Franks' theorem does not tell us the knot types of $a$ and $r$.

# Main Theorem

## Theorem

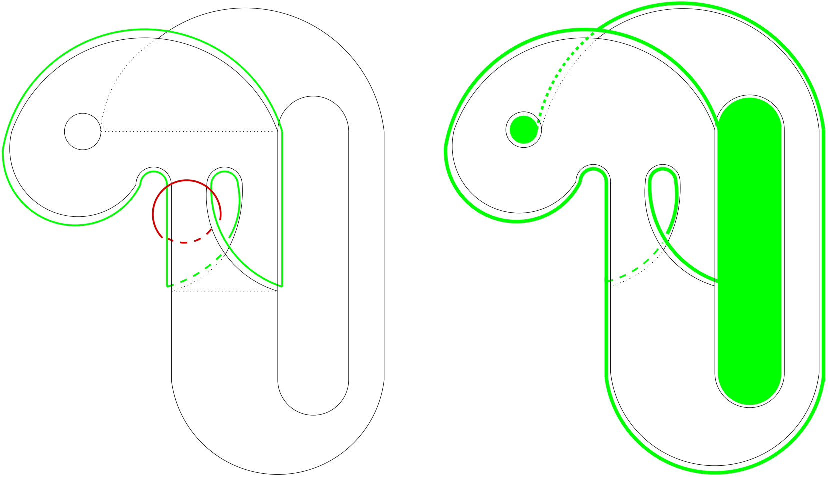
*For every $n \geq 2$ there exists a simple Smale flow where $a$ and $r$ are unknotted and the saddle set has a cross section whose first return map is conjugate to the full $n$-shift.*

## Outline of Proof.

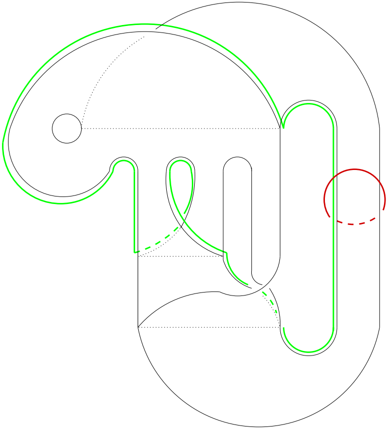
The proof is pictorial and uses induction on $n$.
The induction step is really a two-step.
If the claim holds for an even value of $n$, then it holds for $n+1$.
If the claim holds for an odd value of $n$, then it holds for $n+1$.
The details of the arguments are different.

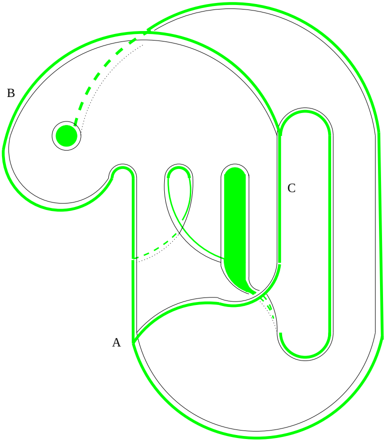
For $n = 2$ the result is known. [S, 2000] We will show examples for $n = 3, 4, 5, 6$ and discuss the induction step. □

# N=3



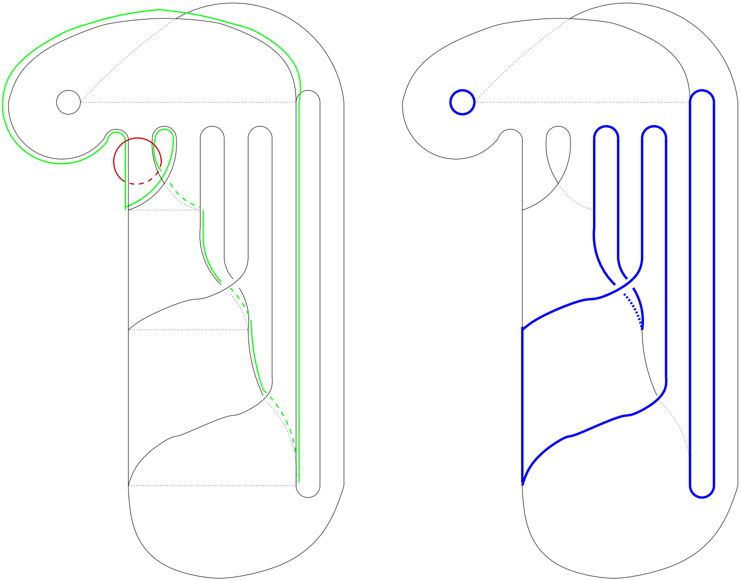
# N=4



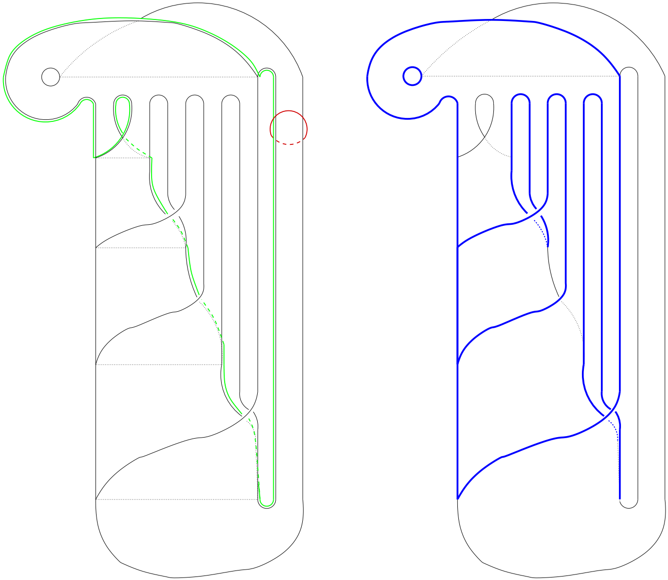



Cap off along loop ABC with $D \times I$.

# N=5



# N=6



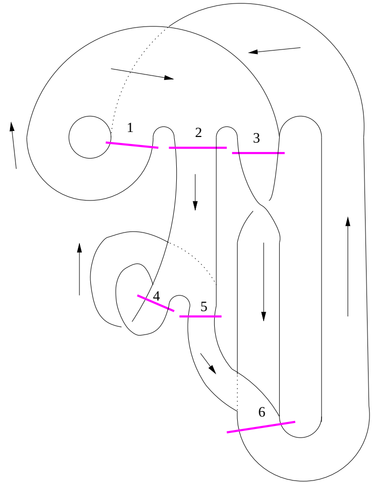
# Incidence and Structure Matrices

Let $D_1, D_2, \ldots, D_n$ be cross sectional disks for Markov partition. Assume the are "small enough" for the definitions below to work.

Incidence Matrix: Let $A = [a_{ij}]$ be given by $a_{ij} = 1$ if there is a orbit directly from $D_i$ to $D_j$ and be zero otherwise.

Structure Matrix: Let $S = [s_{ij}]$ be given by $s_{ij} = \pm a_{ij}$, with "$-$" meaning the first return map is orientation reversing.

# An Example




$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

# Linking Numbers

## Theorem (Franks 1981)

*In a simple Smale flow the unsigned linking number of a and r is the determinant of I minus the structure matrix,*

$$|lk(a, r)| = |\det(I - S)| .$$

## Corollary

*For a simple Smale flow with saddle set a suspension of the n-shift we have*

$$lk(a, r) = \begin{cases} \text{even if } n \text{ is odd,} \\ \text{odd if } n \text{ is even.} \end{cases}$$

# Conjecture

| $l \backslash n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|
| 0 | $N$ | $Y$ | $N$ | $Y$ | $N$ | $Y$ | $N$ | ... |
| 1 | $Y$ | $N$ | $Y$ | $N$ | $Y$ | $N$ | $Y$ | ... |
| 2 | $N$ | $Y$ | $N$ | $Y$ | $N$ | $Y$ | $N$ | ... |
| 3 | ? | $N$ | ? | $N$ | ? | $N$ | ? | ... |
| 4 | $N$ | ? | $N$ | ? | $N$ | ? | $N$ | ... |
| 5 | ? | $N$ | ? | $N$ | ? | $N$ | ? | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

## Conjecture

*All ? = Y.*

## Questions

*What about allowed knot types? Whitehead links?*

# References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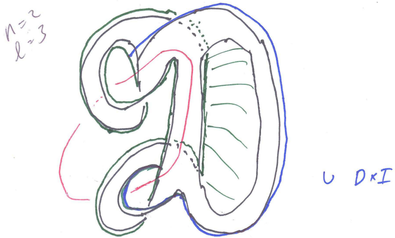
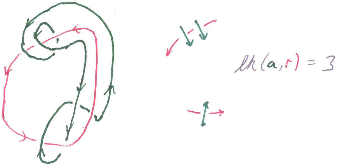
# N=2, L=3 example




$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to$ 2-shift, $n=2$

$I - \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has det 3.