INVARIANTS OF TWIST-WISE FLOW EQUIVALENCE

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Abstract. Flow equivalence of irreducible nontrivial square nonnegative integer matrices is completely determined by two computable invariants, the Parry-Sullivan number and the Bowen-Franks group. Twist-wise flow equivalence is a natural generalization that takes account of twisting in the local stable manifold of the orbits of a flow. Two new invariants in this category are established.

1. Introduction and Background

Square nonnegative integer matrices are used to describe maps on Cantor sets known as subshifts of finite type. Two such *incidence matrices* are *flow equivalent* if their induced subshifts of finite type give rise to topologically equivalent *suspension flows.* The suspension flow is a one-dimension flow obtained by taking the cross product of the Cantor set C with the closed unit interval and gluing $C \times 0$ to $C \times 1$ with a map naturally induced by the original subshift of finite type. *Topologically equivalent* just means there is a homeomorphism between two such flows, taking orbits to orbits, while preserving the flow direction. A matrix A is *irreducible* if for each (i,j) there is an integer n such that the (i,j) entry of A^n is nonzero. In terms of the corresponding subshift and suspension, irreducibility is equivalent to the existence of a dense orbit. Irreducible permutation matrices give rise to flows with a single closed orbit and are thus said to form the *trivial flow equivalence class.* For nontrivial irreducible incidence matrices John Franks has shown that flow equivalence of matrices is completely determined by two computable invariants, the Parry-Sullivan number and Bowen-Franks group. See [10], [1], and [3] or the recent text [8]. Danrun Huang has settled the difficult classification problem arising when the assumption of irreducibility is dropped, [5, 6, 7].

Definition 1.1. Let A be a $n \times n$ nonnegative integer matrix. Then

$$PS(A) = \det(I - A), \text{ and } BF(A) = \frac{\mathbb{Z}^n}{(I - A)\mathbb{Z}^n},$$

are the Parry-Sullivan number and the Bowen-Franks group respectively.

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Remark 1.2. The group $BF(A)$ has infinite order if and only if $PS(A) = 0$. If $BF(A)$ has finite order then its order is given by $|PS(A)|$.

Represent $\mathbb{Z}_2$ by $\{1, t\}$, under multiplication with $t^2 = 1$. Let $A(t)$ be an $n \times n$ matrix with entries of the form $a + bt$, with a and b nonnegative integers. That is A is a matrix over the semigroup ring $\mathbb{Z}^+\mathbb{Z}_2$. Call such a matrix a *twist matrix.*

One interpretation of twist matrices is as follows. Suppose the suspension flow for $A(1)$ is realized as a 1-dimensional basic set $\mathcal{B}$, of saddle type, of a flow on a 3-manifold. For each orbit in $\mathcal{B}$ there is a 2-dimensional local stable manifold, a *ribbon*, if you like. Call the union of such ribbons the *ribbon set*, and denote it by $\mathcal{R}$. Each ribbon is either an annulus, a Möbius band, or an infinity long strip.

Now, $A(1)$ is the incidence matrix for the first return map ρ on the rectangles of a Markov partition, $\{R_1, \ldots, R_n\}$, of a cross section of a neighborhood of $\mathcal{B}$, where $A_{ij}(1)$ is the number of times R_i passes through R_j. If we orient the rectangles then we can let a_{ij} be the number of components of $\rho(R_i) \cap R_j$ where orientation is preserved, and b_{ij} be the number of components where orientation is reversed by the action of ρ. Then $A_{ij}(t) = a_{ij} + b_{ij}t$.

Figure 1 gives an example. The map is just the horseshoe map. A piece of an orbit of a suspension flow is shown. The section of ribbon shown has no net twist. Using the shaded rectangle as a Markov partition the twist matrix is $[1 + t]$. If we use the two smaller rectangles shown the twist martix is $\begin{bmatrix} 1 & t \\ 1 & t \end{bmatrix}$.

Note: From the symbolic point of view the shift map of $[1 + t]$ would be modeled by the edge-shift of a graph with one vertex and two edges e_1 and e_2 (one labeled with a 1, the other with a t). The natural choice for a Markov partition would have two "rectangles": bi-infinite sequences with e_1 in position 1 and bi-infinite sequences with e_2 in position 1. This was the point of view we used in [11].

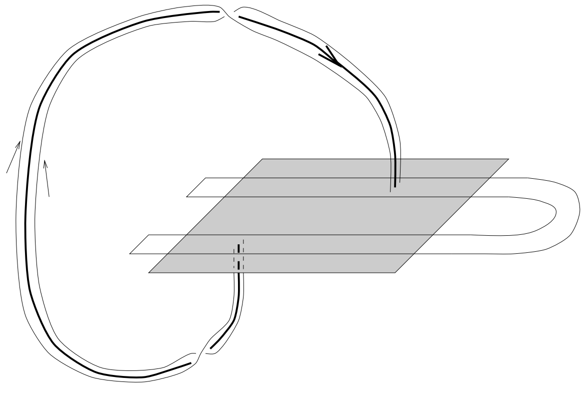

FIGURE 1. A suspension of the map horseshoe map with a ribbon shown

It is not necessary that the manifold be 3-dimensional or that there be only one stable eigenvalue. We only need a means of assigning orientations to rectangles of a Markov partition. We note that if $A(t)$ has only zeros, ones and t's as entries, then $A(-1)$ is the *structure matrix* of [1].

Two ribbon sets are *topologically equivalent* if there is a homeomorphism between them that preserves the flow direction. This leads us to define two twist matrices to be *twist-wise flow equivalent* if they induce topologically equivalent ribbon sets. If A and B are twist-wise flow equivalent we shall write $A \overset{tf}{\sim} B$. It is clear that $PS(A(1))$ and $BF(A(1))$ are invariants in this category. In [11] it is shown that $PS(A(-1))$ is also invariant. The purpose of this paper is to show that $BF(A(-1))$ is too (Theorem 3.1). We develop an additional invariant by using a representation of $\mathbb{Z}_2$ with 2×2 integer matrices (Theorem 3.2). Section 4 gives a topological interpretation of this second new invariant in terms of a double cover of the basic set. In Theorem 3.6 we show that no additional information is gained by using other integer matrix representations of $\mathbb{Z}_2$. Example 4.3 shows that these five invariants are not complete. These results were announced in [12].

Remark 1.3. A *template* is a branched two-manifold with an expanding semi-flow that is used to model hyperbolic invariant sets of flows on three-manifolds. A twist matrix encodes some of the topology of a template whereas the incidence matrix contains only dynamical data. Figure 2 shows a template for the suspended horseshoe map of Figure 1. Invariants of twist equivalence can be seen as part of a program of constructing topological invariants of templates. See [11] for examples and more on this point of view.

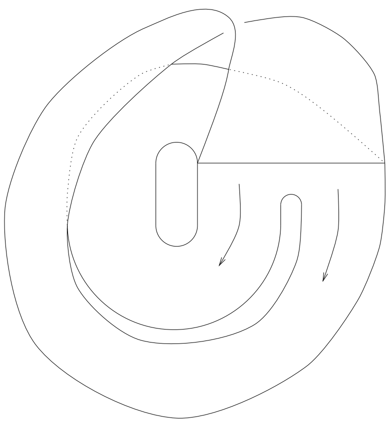

FIGURE 2. Template for horseshoe flow

2. Matrix moves

Twist-wise flow equivalence (or *twist equivalence* for short) is generated by three matrix moves [11]. That is two twist matrices are twist equivalent if and only if there exists a finite sequence of these three moves taking one matrix to the other. The three moves are called, the *shift* move, the *expansion* move, and the *twist* move, and are denoted by $\overset{s}{\sim}$, $\overset{e}{\sim}$ and $\overset{t}{\sim}$, respectively. The first two generate flow equivalence [10]. We define them below.

Shift: $A \overset{s}{\sim} B$ if there exists rectangular matrices R and S, over $\mathbb{Z}^+\mathbb{Z}_2$, such that $A = RS$ and $B = SR$.

Remark 2.1. For square martices over the nonnegative integers the shift move generates the *strong shift equivalence* relation, which can be defined for twist matrices as well. Strong shift equivalence implies *shift equivalence* a weaker relation that we shall not define here. However, for integer matrices shift equivalnce does imply strong shift equivalence, a fact we shall use later. See [8] and also [9].

Expansion: $A \overset{e}{\sim} B$ if $A = [A_{ij}]$ and

$$B = \begin{bmatrix} 0 & A_{11} & \cdots & A_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & A_{21} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} & \cdots & A_{nn} \end{bmatrix}.$$

Twist: $A \overset{t}{\sim} B$ if $A = [A_{ij}]$ and

$$B = \begin{bmatrix} A_{11} & tA_{12} & \cdots & tA_{1n} \\ tA_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ tA_{n1} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

The shift move includes relabelings, so the expansion and twist moves can be done on other "locations" in the matrix. See [11] for geometric motivations.

Another set of matrix moves we shall use are listed below.

1. Exchanging two rows or two columns.
2. Multiply a row or column by -1.
3. Add an integer multiple of one row to another row, or of one column to another column.
4. Delete the i-th row and i-th column if their only nonzero entry is a 1 on the diagonal.

The first three are the standard moves of matrix equivalence; if two $n \times n$ integer matrices are similar over the integers then they are equivalent in this sense. If two square integer matrices, A and B, are related by a finite sequence of these four moves then the associated groups, $\mathbb{Z}^n/A\mathbb{Z}^n$ and $\mathbb{Z}^m/B\mathbb{Z}^m$ are isomorphic. We shall denote equivalence under these four moves by $A \overset{m}{\sim} B$.

Similarity over the integers is denoted by $\overset{\mathbb{Z}}{\sim}$, and similarity over the rationals and reals are denoted by $\overset{\mathbb{Q}}{\sim}$ and $\overset{\mathbb{R}}{\sim}$, respectively.

To condense our exposition we adopt a few conventions for writing matrices. We will sometimes write $[a, b; c, d]$ instead of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or even $abcd$ if no confusion can arise. Diagonal matrices may be written as $\operatorname{diag}(a, b, \ldots, z)$, and similarly for block diagonal matrices. The direct sum of two square matrices, denoted $A \oplus B$, is just the block diagonal matrix $\operatorname{diag}(A, B)$.

3. Main Results

Theorem 3.1. *Let $A(t)$ and $B(t)$ be twist matrices with $A(t) \overset{tf}{\sim} B(t)$. Then $BF(A(-1)) \cong BF(B(-1))$.*

Proof. Suppose $A(t) \overset{s}{\sim} B(t)$. Then $A(-1)$ is shift equivalent to $B(-1)$ over the integers. Hence, $BF(A(-1)) \cong BF(B(-1))$ by Theorem 7.4.17 of [8].

Suppose $A(t) \overset{t}{\sim} B(t)$. But then $I - A(-1)$ can be transformed to $I - B(-1)$ by multiplying the first row and the first column by -1. Hence, $BF(A(-1)) \cong BF(B(-1))$.

Suppose $A(t) \overset{e}{\sim} B(t)$. Let $[A_{ij}] = A(-1)$. Then

$$I - B(-1) = \begin{bmatrix} 1 & -A_{11} & -A_{12} & -A_{13} & \cdots & -A_{1n} \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -A_{21} & 1 - A_{22} & -A_{23} & \cdots & -A_{2n} \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & -A_{n1} & \cdots & \cdots & -A_{nn-1} & 1 - A_{nn} \end{bmatrix} \overset{m}{\sim}$$

$$\begin{bmatrix} 1 & 1 - A_{11} & -A_{12} & -A_{13} & \cdots & -A_{1n} \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -A_{21} & 1 - A_{22} & -A_{23} & \cdots & -A_{2n} \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & -A_{n1} & \cdots & \cdots & -A_{nn-1} & 1 - A_{nn} \end{bmatrix} \overset{m}{\sim}$$

$$\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I-A(-1) & \\ 0 & & & \end{bmatrix} \overset{m}{\sim} I-A(-1).$$

The proof is completed by induction. □

Let $T \neq I$ be a $k \times k$ integer matrix with $T^2 = I$. Thus, $\mathbb{Z}_2 \cong \{I, T\}$. If $A(t) = [a_{ij} + b_{ij}t]$ is an $n \times n$ twist matrix let $A(T)$ be the $nk \times nk$ integer matrix formed from n^2 $k \times k$ blocks given by $A_{ij}(T) = a_{ij}I + b_{ij}T$. Notice that each of the blocks in $A(T)$ commutes with T.

Theorem 3.2. *If $A(t) \overset{tf}{\sim} B(t)$ then $BF(A(T)) \cong BF(B(T))$ and $PS(A(T)) = PS(B(T))$.*

Proof. Suppose $A(t) \overset{s}{\sim} B(t)$. Then there exists matrices $R(t)$ and $S(t)$ such that $A(t) = R(t)S(t)$ and $B(t) = S(t)R(t)$. But then $A(T) = R(T)S(T)$ and $B(T) = S(T)R(T)$. Thus, $A(T)$ is shift equivalent to $B(T)$ over the integers. Hence, $BF(A(T)) \cong BF(B(T))$. Since PS is also a shift invariant $PS(A(T)) = PS(B(T))$.

Suppose $A(t) \overset{t}{\sim} B(t)$. Define $\hat{T}$, a $nk \times nk$ matrix, by $\hat{T} = \operatorname{diag}(T, I, \ldots, I)$. Then $\hat{T}^2 = I$, and $B(T) = \hat{T}A(T)\hat{T}$, where we have used the fact that T commutes with the $aI + bT$ blocks. But also, $I - B(T) = \hat{T}(I - A(T))\hat{T}$. Hence, $BF(A(T)) \cong BF(B(T))$. For the Parry-Sullivan number one gets $PS(B(T)) = \det(I - B(T)) = \det(\hat{T})\det(I - A(T))\det(\hat{T}) = \det(I - A(T)) = PS(A(T))$.

Suppose $A(t) \overset{e}{\sim} B(t)$. Then $B(T)$ can be obtained from $A(T)$ by k expansions, interspersed with relabelings. Hence, $BF(A(T)) \cong BF(B(T))$ by the proof of Theorem 3.1. It is obvious that $PS(A(T)) = PS(B(T))$.

Again, induction completes the proof. □

Remark 3.3. Theorem 3.2 holds true when $T = I$.

Theorem 3.4. *If T_1 is similar to T_2 over $\mathbb{Z}$ then $BF(A(T_1)) \cong BF(A(T_2))$ and if T_1 is similar to T_2 over $\mathbb{R}$ then $PS(A(T_1)) = PS(A(T_2))$. If $T = T_1 \oplus T_2$ then $BF(A(T)) = BF(A(T_1)) \oplus BF(A(T_2))$ and $PS(A(T)) = PS(A(T_1)) \times PS(A(T_2))$.*

Proof. Let T_1 and T_2 be $k \times k$ matrices and let A be $n \times n$. Next assume $T_1 = PT_2P^{-1}$ with P and P^{-1} in $\mathbb{Z}^{k\times k}$ or $\mathbb{R}^{k\times k}$ as needed. Let $\hat{P} = \operatorname{diag}(P, \ldots, P)$ be an $nk \times nk$ matrix. Then $A(T_1) = \hat{P}A(T_2)\hat{P}^{-1}$. From this it is clear that $BF(A(T_1)) \cong BF(A(T_2))$ and $PS(A(T_1)) = PS(A(T_2))$.

If $T = T_1 \oplus T_2$ then by elementary row and column operations (that only involve switching rows and switching columns) one can show that $A(T) \sim A(T_1) \oplus A(T_2)$. The remaining conclusions of the theorem follow. □

Example 3.5. Let $T_1 = [0, 1; 1, 0]$ and $T_2 = [-1, 0; 0, 1]$. Then $T_1^2 = I = T_2^2$, and T_1 is similar to T_2 over $\mathbb{Q}$ but not over $\mathbb{Z}$. Also $T_2 = [-1] \oplus [1]$. Thus, $PS(A(T_1)) =$

$PS(A(T_2)) = PS(A(1)) \times PS(A(-1))$. Further, $BF(A(T_2)) = BF(A(1)) \oplus BF(A(-1))$. But, $BF(A(T_1))$ may contain non-redundant invariant information. We will see in the next section (Example 4.4) that this is indeed the case.

In the theorem below we show that using representations of $\mathbb{Z}_2$ with $n \times n$ integer matrices with $n \geq 3$ yields no additional information.

Theorem 3.6. *Let $\mathcal{T}_n = \{T \in \mathbb{Z}^{n\times n} \mid T^2 = I\}$. If $T \in \mathcal{T}_n$ then there exist nonnegative integers i, j, and k, depending only on T, with $i + j + 2k = n$, such that*

(*i*) $BF(A(T)) = [BF(A(1))]^{\oplus i} \oplus [BF(A(-1))]^{\oplus j} \oplus [BF(A([0,1;1,0]))]^{\oplus k}$,
and,
(*ii*) $PS(A(T)) = [PS(A(1))]^{i+k}[PS(A(-1))]^{j+k}$.

Proof. Let $T \in \mathcal{T}_n$. Hua and Reiner [4, Lemma 1] have shown that

$$T \overset{\mathbb{Z}}{\sim} [1]^{\oplus i} \oplus [-1]^{\oplus j} \oplus [1,0;1,-1]^{\oplus k},$$

for some i, j and k. But $[0,1;1,0] \overset{\mathbb{Z}}{\sim} [1,0;1,-1]$. Thus,

$$T \overset{\mathbb{Z}}{\sim} [1]^{\oplus i} \oplus [-1]^{\oplus j} \oplus [0,1;1,0]^{\oplus k}.$$

It follows that $T \overset{\mathbb{Q}}{\sim} [1]^{\oplus i+k} \oplus [-1]^{\oplus j+k}$, since $[0,1;1,0] \overset{\mathbb{Q}}{\sim} [1] \oplus [-1]$. Now apply Theorem 3.4. $\square$

4. Double cover flows

For a 2-dimensional ribbon set $\mathcal{R}$ place a flow on the boundary with direction parallel to the flow on its core $\mathcal{B}$. Call this the *double cover flow* of $\mathcal{B}$. An incidence matrix ∂A can be constructed from a twist matrix $A(t)$ by replacing each entry $a+bt$ with $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. This amounts to using the matrix representation

$$\mathbb{Z}_2 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

To see why this is so, consider Figure 3. Figure 3a represents the formation of the double cover flow between two elements of a Markov partition, labeled 1 and 2 for simplicity. Partition element 1 becomes two partition elements, 1 and 2, while the original number 2 changes to 3 and 4. The new 1 goes to the new 3, and the new 2 goes to the new 4 under the flow. Thus, if the 12 element of the original twist matrix is 1 it gives rise to the 2×2 block $[1,0;0,1]$ in the incidence matrix of the double cover flow. If it had been a t Figure 3b shows that it would give a $[0,1;1,0]$ block. In general, the ij-th entry of a twist matrix changes to the 2×2 block containing the

$(2i-1, 2j-1)$, $(2i, 2j-1)$, $(2i-1, 2j)$, and $(2i, 2j)$ entries in the incidence matrix of the double cover flow. For example, if we let $A(t) = [1, 1; t, t]$ we get $\partial A = A(0110) =$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

In this case $BF(\partial A)$ is just the trivial group.

We note that John Guckenheimer has developed a concept of double covers of surface diffeomorphisms. See [2, §3].

Notation. When no confusion can arise we shall denote $BF(A(1))$ by BF^+, $BF(A(-1))$ by BF^-, $BF(A(0110))$ by BF^∂, and similarly for the Parry-Sullivan numbers.

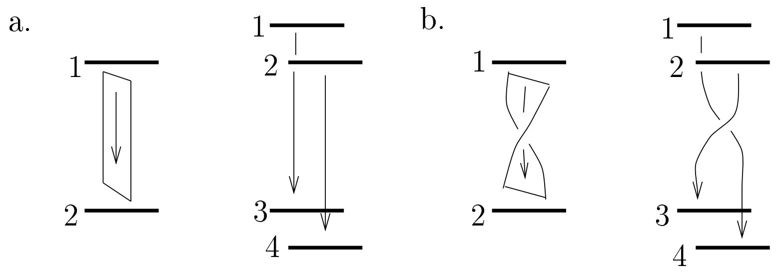

FIGURE 3. Taking the double cover flow.

Example 4.1. For the 2×2 matrix $t111$ we get $PS^+ = PS^- = -1$. Since, $|PS^\pm|$ is the order of $BF^\pm$ (respectively) we get that both $BF^\pm$ groups are trivial groups. For $1t11$ we get $PS^+ = -1$ while $PS^- = 1$, and thus $t111$ and $1t11$ are in different twist-wise flow equivalence classes. The pair of 3×3 matrices, $011t011t1$ and $011t0111t$, are also only distinguished by the sign of PS^-, but here the Bowen-Franks groups are not trivial.

Example 4.2. For $tt11$ we get $PS^+ = -1$ and $PS^- = 1$ as with $1t11$. The double cover invariant yields no additional information since BF^∂ is the trivial group in both cases. But, we have not been able to find a sequence of moves ($\overset{s}{\sim}, \overset{e}{\sim}, \overset{t}{\sim}$) that would show these two matrices to be twist equivalent. Since $tt11$ has a "period one" Möbius band in its ribbon set, and $1t11$ does not, such a sequence would have to include an expansion move, perhaps many.

Example 4.3. The matrix 1111 gives $PS^+ = PS^- = -1$, as did $t111$. Passing to the double cover gains us nothing. Yet, 1111 has an orientable ribbon set while $t111$ does not. So, they cannot be twist equivalent. Thus, our invariants cannot be complete.

Of course one can always take orientability itself as an invariant. It may even be that orientation and the PS and BF invariant are complete. However, my intuition is that this is not the cast. Another such example is given by the pair of 3×3 matrices, 011101111 and 01110111t.

Example 4.4. Now consider $A = [3, 1 + t; 1 + t, 3]$, and $B = [3, 1 + t; 2, 3]$. We get $PS^+ = 0$, $BF^+ = \mathbb{Z} \oplus \mathbb{Z}_2$, $PS^- = 4$, and $BF^- = \mathbb{Z}_2^2$ for both matrices. But $BF^\partial(A) = \mathbb{Z} \oplus \mathbb{Z}_4$ while $BF^\partial(B) = \mathbb{Z} \oplus \mathbb{Z}_2^2$. Thus, A and B are in distinct twist-wise flow equivalence classes.

For additional examples see Table 2 in [12].

We shall now classify all 1×1 twist matrices using only the Parry-Sullivan numbers.

Theorem 4.5. *Let $[a_1+b_1t]$ and $[a_2+b_2t]$ be twist matrices. Then $[a_1+b_1t] \overset{tf}{\sim} [a_2+b_2t]$ if and only if $a_1 = a_2$ and $b_1 = b_2$.*

Proof. Suppose $[a_1 + b_1t] \overset{tf}{\sim} [a_2 + b_2t]$. Then $PS^+([a_1 + b_1t]) = PS^+([a_2 + b_2t])$ implies $a_1 + b_1 = a_2 + b_2$. Similarly $a_1 - b_1 = a_2 - b_2$. Hence, $a_1 = a_2$ and $b_1 = b_2$. The other direction is obvious. □

Consider 2×2 matrices with entries 0, 1, or t, but which are irreducible and nontrivial. Will we divide these matrices into 6 classes:

- $A = \{1111, 1110, 1101, 1011, 1tt1, 1tt0, 0tt1\}$
- $B = \{t111, 111t, ttt1, 1ttt\}$
- $C_1 = \{1t11, 11t1\}$
- $C_2 = \{tt11, t1t1, 1t1t, 11tt, 1t10, 11t0, 0t11, 01t1, t110, ttt0, 011t, 0ttt\}$
- $D = \{t11t, tttt, tt10, t1t0, 01tt, 0t1t\}$
- $E = \{t1tt, tt1t\}$

Within each of these classes one can show that the matrices are twist equivalent by constructing the necessary matrix moves. In Table 1 below we list the invariants for each class. Classes A and B are not distinguished by any of our invariants. Yet their ribbon sets cannot be homeomorphic since the ribbon set for B contains Möbius bands whereas the closed ribbons for A are all annuli, as was also noted in Example 4.3 above. The classes C_1 and C_2 also have the same set of invariants. However, we have not been able to tell if they form a single twist class or not.

Theorem 4.6. *The members of class A are twist equivalent to the 1×1 matrix $[2]$, those in class C_2 to $[1 + t]$, and those in D to $[2t]$. The members of classes B and E are not twist equivalent to any 1×1 twist matrix.*

Proof. We use $tt11 \in C_2$. $\begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$, while, $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = [1 + t]$. Similar computations for representatives of A and D are left to the reader.

By the proof of Theorem 4.5 if a member of class B is twist equivalent to a 1×1 matrix, this matrix is determined by its Parry-Sullivan numbers. In this case we would get the twist matrix [2], but we have shown that this is not possible.

For class E a calculation with its Parry-Sullivan numbers shows that if its members were equivalent to a 1×1 twist matrix it would have to be $[-1+2t]$, which is not a valid twist matrix. ☐

Table 1

Class	PS^+	BF^+	PS^-	BF^-	BF^∂
A	-1	0	-1	0	0
B	-1	0	-1	0	0
C_1	-1	0	1	0	0
C_2	-1	0	1	0	0
D	-1	0	3	$\mathbb{Z}_3$	$\mathbb{Z}_3$
E	-1	0	5	$\mathbb{Z}_5$	$\mathbb{Z}_5$

5. Generalized Bowen-Franks groups?

For strong shift equivalence there is a well known generalization of the Bowen-Franks group. Let $p(s) \in \mathbb{Z}[s]$ with $p(0) = \pm 1$. Let $BF_p(A) = \mathbb{Z}^n / p(A))\mathbb{Z}^n$. Then if A and B are strong shift equivalent incidence matrices $BF_p(A) = BF_p(B)$ See [8, §7.4]. If A and B are twist matrices with $A \overset{t}{\sim} B$ then it is easy to show that $BF_p(A(\pm 1)) = BF_p(B(\pm 1))$, where there are no restrictions on the polynomial. However, we have not been able to carry over this type of generalization to the expansion move.

Proposition 5.1. *Let $p(s) = 1 + ks \in \mathbb{Z}[s]$. If*

$$\mathbb{Z}^n / p(A(\pm 1))\mathbb{Z}^n \cong \mathbb{Z}^{n+1} / p(B(\pm 1))\mathbb{Z}^{n+1},$$

respectively, for all twist matrices A with $A \overset{e}{\sim} B$, then $k = -1$ or 0. Here n is the dimension of A and $n+1$ is the dimension of B.

Proof. Let $A = [11; 11]$ and $B = [011; 100; 011]$. Then $A \overset{e}{\sim} B$. Then $\det[P(A(1))] = 1 + 2k$ while $\det[P(B(1))] = 1 + k - k^2$. But $|1 + 2k| = |1 + k - k^2|$ if and only if $k = -1$ or 0. Hence, the corresponding groups have different orders. ☐

Conjecture 5.2. *For any nontrivial polynomial $p(s) \in \mathbb{Z}[s]$ not equal to $1 - s$ there exists a pair of expansion equivalent twist matrices A and B such that $\mathbb{Z}^n / p(A(1))\mathbb{Z}^n$ and $\mathbb{Z}^{n+1} / p(B(1))\mathbb{Z}^{n+1}$ are not isomorphic. Here n and $n+1$ are the dimensions of A and B respectively.*

It would be nice if a counterexample to this conjecture could be found.

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