POSITIVE BRAIDS WITH A HALF TWIST ARE PRIME

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Abstract

We shall prove that a knot which can be represented by a positive braid with a half twist is prime. This is done by associating to each such braid a smooth branched 2-manifold with boundary and studying its intersection with a would-be cutting sphere.

Keywords: Branched surfaces, knots, positive braids, prime knots.

This paper assumes the reader is familiar with the distinction between prime and composite knots, the fact that knots can be braided and the geometric proof that torus knots are prime. These can be found in [2].

A knot is a *positive braid* if it can be represented as a braid, all of whose crossing are of the same type. It has been conjectured that positive braids with a full twist are prime [4, Problem 18.1]. The conjecture was motivated by the study of Lorenz knots, which arise in many dynamical systems. They are known to be positive braids with a full twist [1] and Bob Williams had shown that they are prime [6]. Here we prove a slightly stronger result, that positive braids with a half twist are prime. After announcing our result we learned that another researcher had also obtained it as a special case of a more general theorem using different methods [3]. We hope that our proof techniques are of interest.

Theorem. Positive braids with a half twist are prime.

Proof. Let k be a positive braid with a half twist. We associate to k a branched surface, $B = B(k)$. The surface is a Möbius band M, with half disks or *tabs* attached along their diameters. There is one tab for each crossing of k that is not a part of the half twist. The attachings are smooth and all

tabs come into the Möbius band from the same direction. The smoothness is needed so that B will have a well defined normal bundle.

We can find in B a knot of knot type k using one tab for each crossing that is not part of the half twist. Figure 1 gives an example. The knot is a (2,5)-torus knot presented on three strands. The embedding is piecewise smooth with one cusp point per branch line.
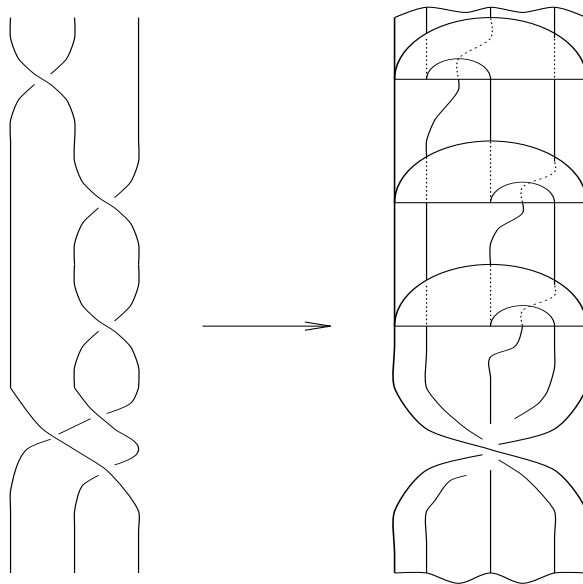

Figure 1: The knot k is taken to the branched surface B. One connects the bottom and top of the figures to form a closed braid and the corresponding branched surface.

Now suppose that k is a composite knot. Then there is a 2-sphere S that factors k. We can take S to be transverse to B. Further, we can require them to be perpendicular. Let $I = B \cap S$. Let β be the union of the branch lines of B. Let $\delta = \partial B \cup \beta$. Divide I into segments whose end points are $I \cap \delta$. We also divide δ in segments using the end points of β.

We assume that the number of segments in I is minimal. This will allow us to construct an algorithm which traces out a path in I with infinitely many segments, contradicting transversality. Hence, the theorem will be proved.

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Minimality tells us that there are no trivial loops which miss β or arcs that connect a segment of ∂B to itself in I. (We regard the trivial loops as segments.) Also, segments of I that connect one segment of ∂M to another (without meeting β) would be pierced by the knot more than once from the same side (here we think of k as being oriented). Next we list the remaining types of segments that could be in I.

- Same branch segments:

  - $\cup$-joints connect two points on the same branch line from below.
  - $\cap$-joints connect two points on the same branch line from above.

- Branch to edge segments:

  - $\lfloor$-joints connect a branch point to a point on ∂M below and to the right.
  - $\rfloor$-joints connect a branch point to a point on ∂M below and to the left.
  - $\lceil$-joints connect a branch point to a point on ∂M above and to the right.
  - $\rceil$-joints connect a branch point to a point on ∂M above and to the left.
  - $|$-joints connect a branch point to a point on $\partial B - \partial M$.

- Branch to branch segments connect one branch line to other (only one type).

Note: the notions of right & left and above & below are well defined in a neighborhood of any branch line. Each branch line can be oriented and the orientations made consistent by using the using the crossing of k. We take the tangents going towards the branche lines to be pointing downward.

Lemma 1. If $p \in I \cap \beta$ then either p is the lower end point of a branch to branch segment or it is the right end point of a $\cap$-joint.

We shall defer the proof of lemma 1 until later.

The algorithm: Pick a point p in $I \cap \beta$. Such a point exists since segments that miss β where ruled out. Choose one of the two segments meeting p from above according to the following rules: If possible choose a branch to branch segment and repeat the algorithm from its other end point. If not, then at least one of the two segments is a $\cap$-joint whose other end point is to our left. Choose it and repeat from this point.

Lemma 1 guarantees that the algorithm is valid and gives a path P which misses ∂B. It only remains to show that P has no loops, for this will imply that P has infinitely many segments. This is done in lemma 2.

Lemma 2. There are no loops in P.

Proof. Consider a unit normal bundle to P. Think of it as a ribbon glued down along P. If P has a loop L, we get a closed ribbon R. Since the sphere is normal to B, we can place R in the sphere. Hence R is untwisted. If L contains no $\cap$-joints then the twist of R is given by

$$\mathrm{Twist}(R) = \frac{3}{2}n - 1,$$

where n is the number of times L goes around on M. But this is never zero, so L has one or more $\cap$-joints. We study how the $\cap$-joints contribute to $\mathrm{Twist}(R)$.

When we travel along a $\cap$-joint we are going from right to left. When we get to the other end point and proceed upwards from there, a half twist is created. When we go from front to back we get a $+\frac{1}{2}$, which only makes matters worse. If we go from back to front we do get a $-\frac{1}{2}$, but in order to form a loop we must get back onto M. Thus we encounter a back to front $\cap$-joint and contribute nothing to the twist. See figure 2. If we let f be the number of front $\cap$-joints and b be the number of back $\cap$-joints, then

$$\mathrm{Twist}(R) = \frac{3}{2}n - 1 + \frac{1}{2}(f - b).$$

But, $f \geq b \Rightarrow \mathrm{Twist}(R) > 0$. Hence there are no loops in P and the proofs of lemma 2 and the theorem are complete. $\square$

Note: Our twist formula is taken from Lemma 5.6 of [6] and is only valid when all the crossings of L are positive, which they are, and when L

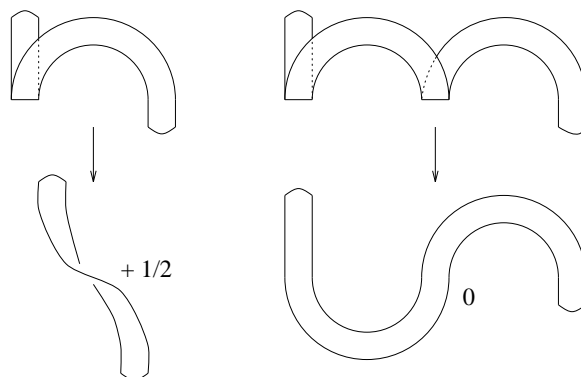is unknotted, which it must be. See section 3 of [5] for a discussion of these points.

Figure 2: We have actually drawn tangent bundles, which have the same twist as the normal bundles but are easier to visualize.

Proof of lemma 1. Consider a $\cap$-joint. Label its left and right end points p and q respectively. We may take it to be inner most. This implies that I does not meet the arc in β bounded by p and q. Call this arc $\overline{pq}$.

If k does not meet $\overline{pq}$ then we can deform S so as to push the $\cap$-joint through the branch line and reduce the number of segments without gaining or losing intersection points between the knot and the sphere. That is we have a new transverse cutting sphere with fewer segments in I. See figure 3.

However, we can say more. If the only points in $k \cap \overline{pq}$ come from arcs of the knot which hit the $\cap$-joint, then we can do the same move as before; the single intersection point is dragged along with the deformation of S. The same is true if the only point in $k \cap \overline{pq}$ is a cusp point. See figure 4. We summarize this discussion by saying: *A $\cap$-joint must be guarded by an arc of k from the opposite branch.*

We will now show that there are no $\rbrack$-joints in I. Suppose that there is one and let its end point on β be p and the left end point of this component of β be e. Call the open arc from e to p, $\overline{ep}$. We assume the $\rbrack$-joint to be left most. If I meets $\overline{ep}$ there would have to be a $\cap$-joint on the Möbius band which cannot be guarded. This is because, the only way to guard the $\cap$-joint
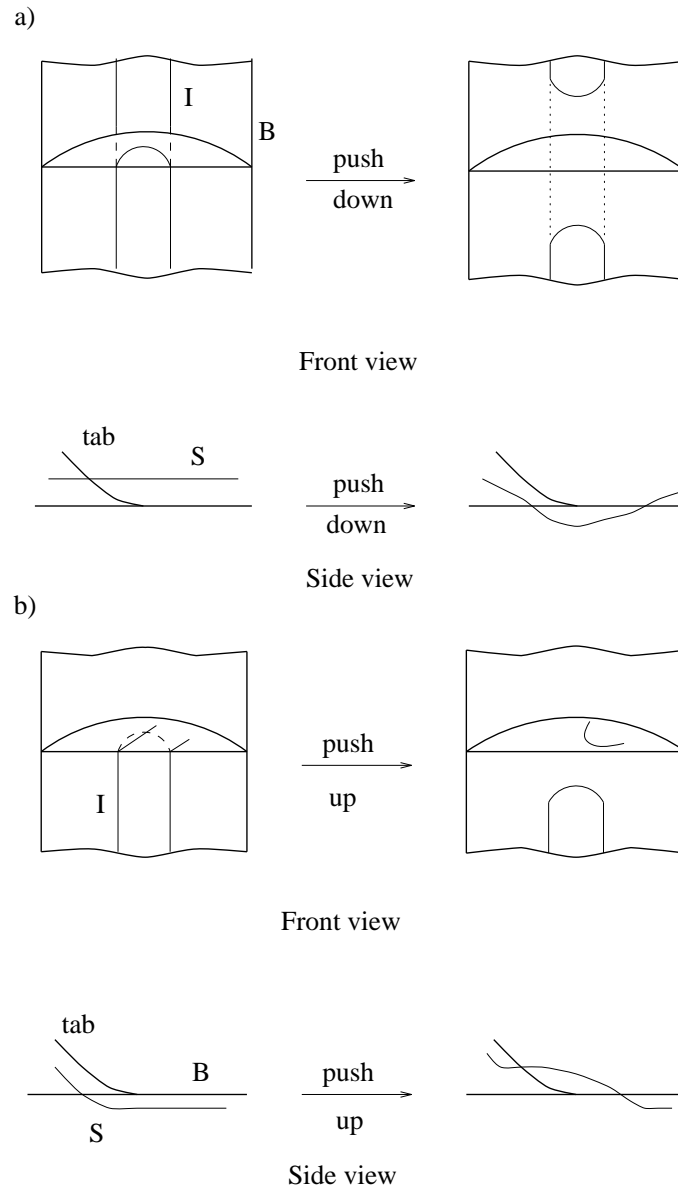
a)

Front view

Side view

b)

Front view

Side view

Figure 3: a) Removing an unguarded front ∩-joint. b) Removing an un-guarded back ∩-joint. c) Guarded ∩-joints. (Next page.)
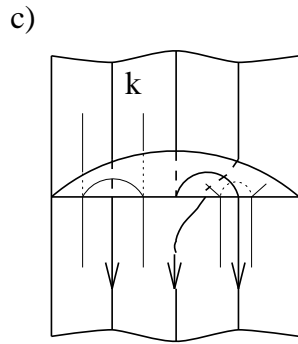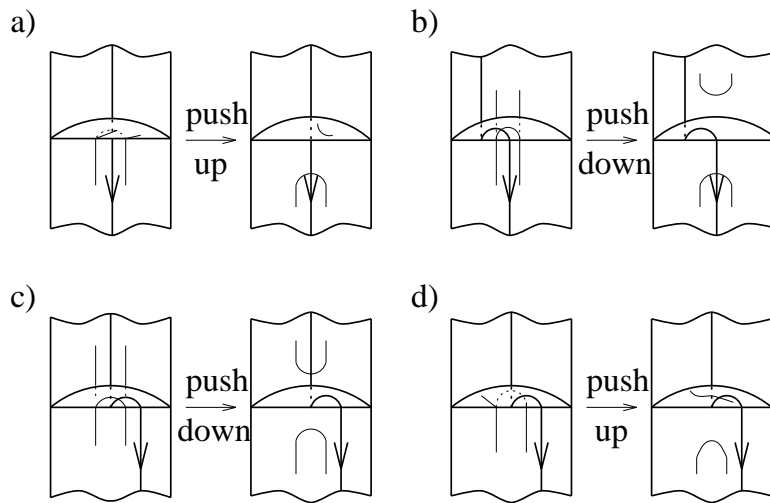
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c)

Figure 3: Continued from previous page

Figure 4: Removing unguarded $\cap$-joints which meet the knot k

would force k to cross the $\rceil$-joint twice from the same side. But then k would pierce S twice from the same side. See figure 5.

Now, since k can meet our $\rceil$-joint at most once, we only have three cases: k misses the joint and hence misses $\overline{ep}$, k meets $\overline{ep}$ at a cusp point, or k meets $\overline{ep}$ transversely. In each case, we can deform S so as to reduce the number
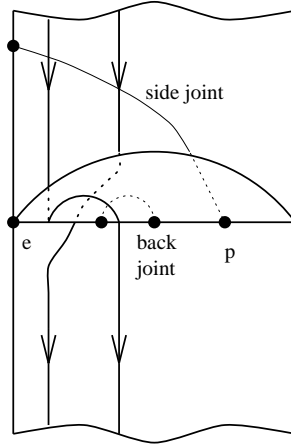
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Figure 5: Double piercing of S from same side

of segments in I. See figure 6. We might say that it is impossible to guard a $\rceil$-joint. While we cannot rule out $\lceil$-joints, it should be clear that they must be guarded by an arc of k on the corresponding tab.

Now, to prove the lemma, let $p \in I \cap \beta$, and assume the lemma is false. Then one of the three cases below must occur, yet each leads to a contradiction.

CASE 1: Suppose p is the common left end point of two $\cap$-joints. We assume the structure is inner most. Then I must miss the interior of the arc of the branch line connecting the end points of the shorter $\cap$-joint.

If the right end points are equal, it is impossible to guard both $\cap$-joints without puncturing the sphere twice from the same side.

Suppose the that the front $\cap$-joint's right end point falls short of the back $\cap$-joint's. In order to guard the front $\cap$-joint a single arc of k must pass under it and hence pierce the other $\cap$-joint. We see in figure 7a that we can still deform S so as to reduce the number of segments even though the $\cap$-joint was guarded. The joint wasn't *guarded well enough.*

If the back $\cap$-joint is the shorter than we must have the over-crossing arc of k to guard it, but again we just can't guard it well enough. This is shown in figure 7b.
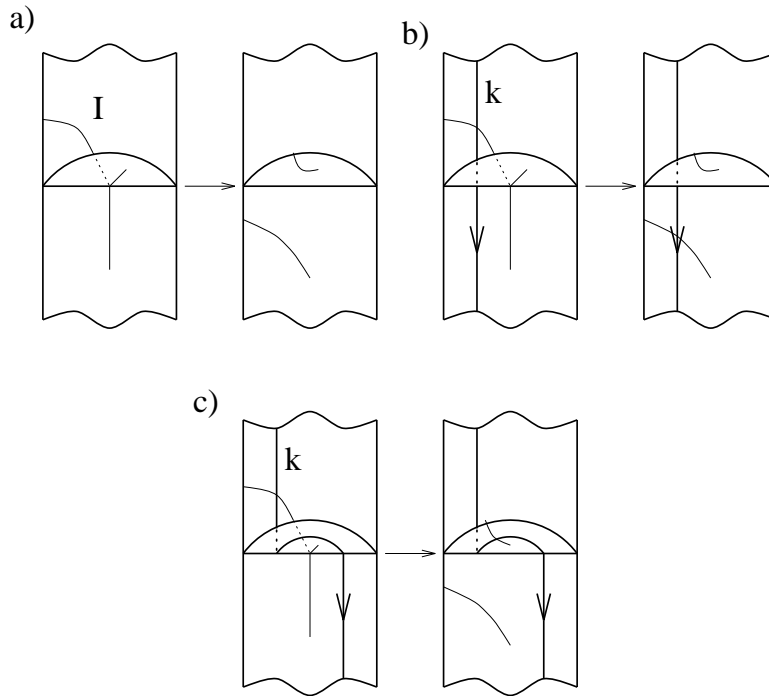
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Figure 6: Removing edge-joints

CASE 2: Suppose a $\lceil$-joint meets the left end point of a $\cap$-joint at p. Then the only way to guard the $\cap$-joint forces k to pierce the $\lceil$-joint. Figure 8 shows how to deform S.

CASE 3: Suppose a $|$-joint and a $\lceil$-joint meet at p. We assume this is the right most such case. If I meets the open interval in β from p to the right end point, then one of the combinations of cases 1 or 2 must occur. Thus, I misses this interval.

Now the only way to guard the $\lceil$-joint is by the arc of k that comes up from the cusp point onto the tab. However, the cusp point is to the right of p, since otherwise the $\lceil$-joint is pierced twice by k from the same side. Just draw the picture. But now k pierces the $|$-joint. The reader can check that as before, the $\lceil$-joint isn't guarded well enough. See figure 9. $\square$
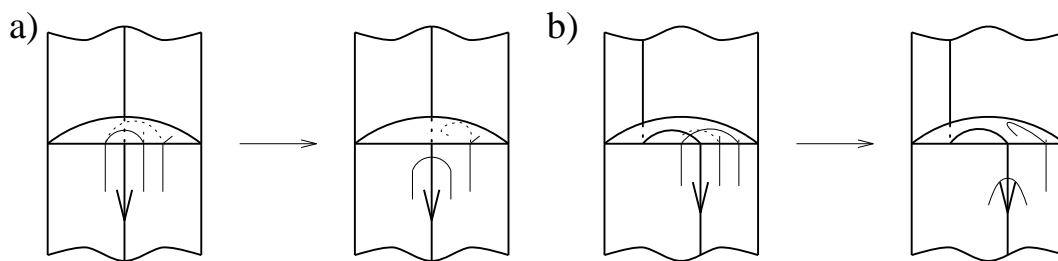
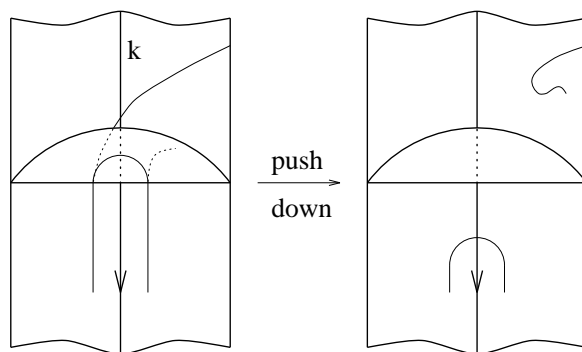Figure 7: Removing ∩-joints that double back
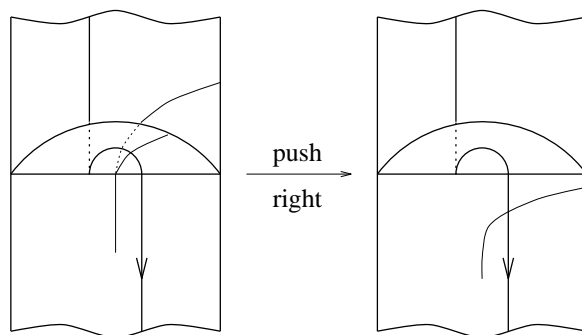

Figure 8: Case 2

Figure 9: Case 3

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