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The linking homomorphism of one-dimensional minimal sets ☆

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Abstract

We introduce a way of characterizing the linking of one-dimensional minimal sets in three-dimensional flows and carry out the characterization for some minimal sets within flows modelled by templates, with an emphasis on the linking of Denjoy continua. We also show that any aperiodic minimal subshift of minimal block growth has a suspension which is homeomorphic to a Denjoy continuum.

MSC: primary 37D45; secondary 37B10

Keywords: Minimal set; Link; Template

1. Introduction

A *flow* is a continuous group action ϕ of $(\mathbf{R}, +)$ on a space X. If f is a continuous $\mathbf{Z}$ or $\mathbf{R}$ action on X, then a closed set $M \subset X$ is a *minimal set* of f if M is invariant but contains no proper, non-empty, closed set which is also invariant under the action. This is equivalent to requiring that the f-orbit of each point of M be dense in M. The simplest one-dimensional minimal sets of flows are periodic orbits, and the linking of periodic orbits in three-dimensional flows has been well-studied; see [4,5,14]. We broaden the perspective and introduce a way of characterizing the linking of one-dimensional minimal sets in three-

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dimensional flows. With an embedding of two minimal sets M and M' in three space we associate a homomorphism $\check{H}_1(M') \to \check{H}^1(M)$ from Čech homology to Čech cohomology with integer coefficients, the *linking homomorphism*. In the case of circular minimal sets, the standard linking number of the embedding represents this homomorphism. In the more general case this linking homomorphism is not necessarily represented by an integer and depends on the structure of the groups $\check{H}_1(M')$ and $\check{H}^1(M)$.

We shall examine the linking homomorphism for minimal sets having minimal block growth in the Lorenz template derived from the full shift on two symbols (see Section 5), which includes the minimal sets derived from Sturmian sequences. The linking homomorphism in this case can be represented by a 2×2 integer matrix, the Smith normal form of which is an invariant of the embedding. In the process we shall show that each such minimal set is homeomorphic to a Denjoy minimal set D_α. Moreover, we shall show that given any D_α from among the uncountably many topologically distinct Denjoy minimal sets (see [12,3]), the union of all homeomorphic copies of D_α in such a template is dense in the template, extending the results of [6]. Since these templates model C^∞ flows in three-space, this is a significant extension of a result of Knill [19], which is of interest since the structure of the Denjoy minimal set prevented Schweitzer's counterexample [26] to the Seifert conjecture from being smoother than $C^{1+\delta}$. (Later Harrison was able to improve this to $C^{2+\delta}$ [18].) Similar observations apply to the general class of isolated and nearly isolated examples examined in [17]. One sees that the behavior of these minimal sets is significantly different when not isolated and that they can interact in complicated ways. (The C^ω example of [20] is two-dimensional.)

For a space X, a metric space M is said to be X-like if for every $\varepsilon > 0$ there is a map $f_\varepsilon : M \to X$ satisfying

$$\operatorname{diam}\bigl(f_\varepsilon^{-1}(x)\bigr) < \varepsilon \quad \text{for all } x \in X.$$

Solenoids and circles are examples of circle-like, one-dimensional minimal sets of flows. In flows in three space one can frequently enclose circle-like minimal sets in tubes homeomorphic to a solid torus. Gambaudo et al. [13] used tubes enclosing minimal sets to define a sort of ergodic linking number. However, such a tube, when retracted to an essentially embedded central circle, provides a natural map to a circle that will be an ε-map when the cross-sectional diameters of the tube do not exceed ε. Hence, if a minimal set is not circle-like, one cannot expect to model the minimal set arbitrarily well with a tube homeomorphic to a solid torus.

With $S^1 \vee S^1$ denoting the wedge of two circles, Denjoy minimal sets are not circle-like, but instead are $(S^1 \vee S^1)$-like, as shown in [3] where the Denjoy minimal set D_α corresponding to the irrational number α is represented as the inverse limit of an inverse sequence of $(S^1 \vee S^1)$'s, the projections of D_α onto the factor $S^1 \vee S^1$ spaces providing the ε-maps. And D_α is not circle-like; otherwise, D_α would be the inverse limit of circles [22] and (by the continuity of Čech cohomology) $\check{H}^1(D_\alpha)$ would then have torsion-free rank one or less, but (as we shall see) $\check{H}^1(D_\alpha)$ is isomorphic to $\mathbf{Z}^2$ and so has torsion-free rank two; see [12] for a discussion of torsion-free rank. Generally, the linking homomorphism is a more appropriate way of characterizing the linking of minimal sets which are not circle-like.

2. The linking homomorphism

Many one-dimensional minimal sets have trivial singular (co)homology (denoted H_1 and H^1) but telling Čech (co)homology. (We shall always use integer coefficients.) When M is not a periodic orbit, any map $S^1 \to M$ must be inessential since the image must be contained in a path component and so must be an arc or a point. Solenoids, for example, have trivial H_1 and H^1, but their one-dimensional Čech cohomology is sufficient for a topological classification [23]. For a Denjoy minimal set D_α there is an inverse limit representation [3]

$$S^1 \vee S^1 \xleftarrow{f_1} S^1 \vee S^1 \xleftarrow{f_2} S^1 \vee S^1 \xleftarrow{f_3} \cdots D_\alpha$$

where the bonding maps f_i depend on the continued fraction expansion of α, but where each f_i (independent of α and i) induces isomorphisms of $H_1(S^1 \vee S^1)$ and $H^1(S^1 \vee S^1)$ which can be represented by matrices in $SL(2, \mathbf{Z})$. Thus, by continuity, $\check{H}_1(D_\alpha)$ and $\check{H}^1(D_\alpha)$ are both isomorphic to $\mathbf{Z}^2$ while having trivial H_1 and H^1. For this reason we use Čech (co)homology.

While we are primarily interested in the linking of minimal sets, we shall define the linking homomorphism for two disjoint, one-dimensional, compact subsets $M, M' \subset \mathbf{R}^3 \subset S^3$. Alexander duality provides an isomorphism

$$A\colon\ \check{H}^1(M) \approx H_1\big(\mathbf{R}^3 - M\big)$$

where

$$\check{H}^1(M) = \varinjlim\big\{H^1(U) \mid U \text{ is a neighborhood of } M\big\}$$

and the system is directed by reverse inclusion and the associated homomorphisms are induced by inclusion. To couch everything in terms of Čech (co)homology, we utilize the isomorphism (see [8, VIII;13.17])

$$\iota\colon\ H_1\big(\mathbf{R}^3 - M\big) \approx \check{H}_1\big(\mathbf{R}^3 - M\big)$$

known to exist since $\mathbf{R}^3 - M$ is a manifold. Here $\check{H}_1$ is given by taking the inverse limit of the dual to the direct sequence used to define $\check{H}^1$. With $j : M' \hookrightarrow \mathbf{R}^3 - M$ denoting the inclusion, the *linking homomorphism* $\Lambda : \check{H}_1(M') \to \check{H}^1(M)$ is given by

$$\Lambda : \check{H}_1\big(M'\big) \xrightarrow{j_*} \check{H}_1\big(\mathbf{R}^3 - M\big) \overset{(\iota)^{-1}}{\approx} H_1\big(\mathbf{R}^3 - M\big) \overset{A^{-1}}{\approx} \check{H}^1(M).$$

3. The linking of Denjoy minimal sets

Denjoy continua form a natural class of minimal sets to which the linking homomorphism applies well. Given two minimal sets M and M' in $\mathbf{R}^3$ and homeomorphisms $h : M \to D_\alpha$ and $h' : M' \to D_{\alpha'}$, we have the inverse limit representations as indicated above and corresponding projections $p_i\,(p_i') : D_\alpha(D_{\alpha'}) \to S^1 \vee S^1$ onto the factor spaces. Since $S^1 \vee S^1$ is an ANR, each of the maps $p_i \circ h$ and $p_i' \circ h'$ extends to a neighborhood U_i or U_i', which (by using finitely many flowbox neighborhoods covering M and M') may be chosen to homotopically retract to a copy of $S^1 \vee S^1$. Since the bonding maps induce isomorphisms on (co)homology, the homomorphism

$$H^1(U_i) \to \check{H}^1(M) = \lim_{\to}\{H^1(U) \mid M \subset U\}$$

given by identifying $H^1(U_i)$ with its occurrence in the direct system defining $\check{H}^1(M)$ is thus in an isomorphism. Similar observations apply to $\check{H}_1(M)$ and M'. Thus, choosing such neighborhoods U_i and U_i' disjoint, we have the following scenario:

$$\begin{array}{ccccccc} \Lambda : \check{H}_1(M') & \xrightarrow{j_*} & \check{H}_1(\mathbf{R}^3 - M) & \overset{(\iota)^{-1}}{\approx} & H_1(\mathbf{R}^3 - M) & \overset{A^{-1}}{\approx} & \check{H}^1(M) \\ \Big\| \wr & & & & & & \Big\| \wr \\ \lambda : H_1(U_i') & \longrightarrow & & & & & H^1(U_i) \end{array}$$

In the cases we shall examine, there is a Mayer–Vietoris decomposition $U_i = A \cup B$ and $U_i' = A' \cup B'$ with each of the sets A, B, A' and B' homotopically equivalent to a circle. Associated with each of these four sets (oriented to go with the flow) is a basis element for (co)homology which admit isomorphisms

$$H_1(U_i') \approx \mathbf{Z}^2 \quad \text{and} \quad H^1(U_i) \approx \mathbf{Z}^2$$

associating $H_1(A') \sim (1, 0)$ and $H_1(B') \sim (0, 1)$ and similarly for U_i. Then the homomorphism λ can be represented by a 2×2 integer matrix $\lambda(M, M')$ which then also represents Λ. Of course, $\lambda(M, M')$ depends on our choice of neighborhoods, amounting to a choice of bases for $\check{H}_1(M')$ and $\check{H}^1(M)$.

Recall that two integer matrices L and L' are *equivalent* if there are matrices X and Y invertible over $\mathbf{Z}$ with $L' = XLY$. By a theorem of Smith, any integer matrix is equivalent to exactly one matrix having entries $m_1, \ldots, m_k$ only along its leading diagonal and satisfying the condition that m_i divides m_{i+1} for $i = 1, \ldots, k - 1$. This uniquely determined diagonal matrix is known as the *Smith normal form.* Thus, the Smith normal form and absolute value of the determinant of $\lambda(M, M')$ are invariants of Λ.

To calculate $\lambda(M, M')$, observe that the first entry of $\lambda(M, M')$ represents the standard linking number of the circles to which A and A' homotopically deform since following the diagram for this entry yields the standard linking homomorphism. Hence, all the standard techniques to calculate this linking number apply; see [25] for a list of such techniques, the most practical of these techniques perhaps being the counting of under/over crossings of a regular projection. Ordinarily the sign of the linking number is considered irrelevant and is regarded as positive in all events, which can always be achieved by changing the choice of basis element in one of the groups. We must however choose consistent orientations in calculating the entries and so some of the entries may not be positive. All four entries are then obtained by calculating the appropriate linking numbers. Reversing the roles of M and M' and using the same pair of neighborhoods to calculate the matrix representation of Λ then transposes the matrix since the linking number of circles is unchanged by reversing the roles of the circles.

In a similar way, one can compute the linking homomorphism between a periodic orbit and a Denjoy continuum. In this case the homomorphism will be represented by a 1×2 matrix or its transpose. In Section 6 we shall investigate the linking of Denjoy minimal sets within the Lorenz template.

4. Minimal sets having minimal block growth

We now turn to the problem of identifying Denjoy continua as minimal sets of well-studied flows by determining which classes of minimal subshifts of the full shift on finitely many symbols are Denjoy continua in their suspended flows. This will allow us then to measure the linking of Denjoy minimal sets as they occur in some natural settings.

We follow the presentations in [16,24] to define the minimal *Sturmian subshift* of the full shift on two symbols (Ω,σ) corresponding to $\alpha\in[0,1]-\mathbf{Q}$. With $\pi:\mathbf{R}\to\mathbf{R}/\mathbf{Z}=S^1$ denoting the quotient map, let $\rho_\alpha:S^1\to S^1$ be the rotation given by $\pi(t)\mapsto\pi(t+\alpha)$ and let $A^-=\pi([0,\alpha))$ and $A^+=\pi((0,\alpha])$. For $t\in\mathbf{R}$, define the sequences $\mathbf{t}^+=\langle t_n^+\rangle_{n\in\mathbf{Z}}$ and $\mathbf{t}^-=\langle t_n^-\rangle_{n\in\mathbf{Z}}$ by

$$t_n^+=\begin{cases}0 & \text{if } \rho_\alpha^n\big(\pi(t)\big)\notin A^+,\\ 1 & \text{if } \rho_\alpha^n\big(\pi(t)\big)\in A^+,\end{cases}$$

and

$$t_n^-=\begin{cases}0 & \text{if } \rho_\alpha^n\big(\pi(t)\big)\notin A^-,\\ 1 & \text{if } \rho_\alpha^n\big(\pi(t)\big)\in A^-.\end{cases}$$

Then $\Omega_\alpha=\{\mathbf{t}^+\mid t\in\mathbf{R}\}\cup\{\mathbf{t}^-\mid t\in\mathbf{R}\}$ is a minimal set of (Ω,σ). We see that $\mathbf{t}^+=\mathbf{t}^-$ for $t\notin\{n\alpha\mid n\in\mathbf{Z}\}$ and that for $n\in\mathbf{Z}$ and $t=n\alpha$, the shift orbits $\sigma^m(\mathbf{t}^+)$ and $\sigma^m(\mathbf{t}^-)$ approach each other asymptotically as $m\to\pm\infty$, meaning $\lim_{m\to\pm\infty}d(\sigma^m(\mathbf{t}^+),\sigma^m(\mathbf{t}^-))=0$. Moreover, these are the only such orbits. Now define $f:\Omega_\alpha\to S^1$, by

$$f\big(\mathbf{t}^+\big)=f\big(\mathbf{t}^-\big)=\pi(t).$$

Then $f^{-1}(f(x))=x$ provided

$$x\notin\big\{\mathbf{t}^+\mid t=n\alpha,\ n\in\mathbf{Z}\big\}\cup\big\{\mathbf{t}^-\mid t=n\alpha,\ n\in\mathbf{Z}\big\},$$

and for

$$x\in\big\{\mathbf{t}^+\mid t=n\alpha,\ n\in\mathbf{Z}\big\}\cup\big\{\mathbf{t}^-\mid t=n\alpha,\ n\in\mathbf{Z}\big\}$$

$f^{-1}(f(x))$ is a two point set. What is more, f provides a semiconjugacy (homomorphism): $\rho_\alpha\circ f=f\circ\sigma|_{\Omega_\alpha}$.

Let us recall the construction of the standard α-Denjoy homeomorphism of S^1. Starting with a single point, say $\pi(0)\in S^1$, one replaces the orbit $\{\rho_\alpha^n(\pi(0))\}_{n\in\mathbf{Z}}$ with a sequence of intervals $\rho_\alpha^n(\pi(0))\rightsquigarrow I_n$ with lengths going to 0 as $n\to\pm\infty$ to obtain a homeomorph of S^1, say S'; see, e.g., [26]. Then by mapping each interval I_n homeomorphically onto I_{n+1} in an orientation preserving way and by mapping all other points of S' to the point determined by ρ_α, we obtain a Denjoy homeomorphism $\delta_\alpha:S'\to S'$ which has a unique minimal set $\mathcal{D}_\alpha$, the Cantor set formed by taking the complement of the interior of the intervals I_n, $n\in\mathbf{Z}$. We denote the points of $\mathcal{D}_\alpha-\bigcup_{n\in\mathbf{Z}}I_n$ by the point $\pi(t)$ of S^1 from which it was derived and we label the interval I_n so that it goes from the point a_n to the point b_n as we follow the orientation. We then have a natural homeomorphism h: $\Omega_\alpha\approx\mathcal{D}_\alpha$ given by

$$h\big(\mathbf{t}^+\big)=h\big(\mathbf{t}^-\big)=\pi(t)\quad\text{for } t\neq n\alpha$$

and for $t = n\alpha$

$$h(\mathbf{t}^-) = a_n \quad \text{and} \quad h(\mathbf{t}^+) = b_n,$$

and h also provides a conjugacy of $\sigma|_{\Omega_\alpha}$ to the restriction of the Denjoy homeomorphism to $\mathcal{D}_\alpha$. By taking the suspension of $\sigma|_{\Omega_\alpha}$ we then obtain a one-dimensional minimal flow which is topologically conjugate to the standard Denjoy flow on D_α.

In what follows, (Σ, σ) denotes the full shift on an alphabet of finitely many symbols. With $P(S, n)$ denoting the number of distinct n-blocks that occur in $S \subset \Sigma$, a minimal subshift $(S, \sigma|_S)$ satisfies $P(S, n) = n + 1$ for all n if and only if $S = \Omega_\alpha$ for some α by the results of [7]. The property $P(S, n) = n + 1$ is not a conjugacy invariant of subshifts; however, the property $P(S, n) \leqslant n + K$ for some fixed K and all n is a conjugacy invariant of minimal subshifts, and any minimal subshift satisfying this condition is said to have *minimal block growth* [24]. Clearly any periodic or Sturmian minimal set has minimal block growth, but the class of aperiodic minimal sets having minimal block growth includes some minimal sets which are not conjugate to any Sturmian minimal set. And yet, the following does hold.

Theorem 1. *A one-dimensional minimal set obtained from the suspension of an aperiodic minimal subshift $(S, \sigma|_S)$ is homeomorphic to a Denjoy continuum if $(S, \sigma|_S)$ is of minimal block growth.*

Proof. As shown in [24, 4.2], any minimal set $(S, \sigma|_S)$ of minimal block growth is obtained from a Sturmian by the composition of a sliding block code and a substitution. A sliding block code yields a conjugacy of symbolic systems, so it is clear that two symbolic systems related by a sliding block code have homeomorphic suspensions. Recall that a homeomorphism $h : X \to X$ is *totally minimal* provided that h^n is minimal for each positive integer n. In [24, 4.1] it is shown that a totally minimal subshift of minimal block growth is conjugate to a Sturmian subshift via a sliding block code.

A substitution of length N associates to each symbol a in the alphabet $\mathcal{A}$ of the original symbolic system an N-block $\theta(a)$ from an alphabet $\mathcal{B}$. This then induces a map of bisequences of $\mathcal{A}$ to those of $\mathcal{B}$:

$$\ldots x_{-2}x_{-1} \,.\, x_0x_1 \ldots \stackrel{\theta}{\longmapsto} \ldots \theta(x_{-2})\theta(x_{-1}) \,.\, \theta(x_0)\theta(x_1) \ldots.$$

If a substitution of length N is applied to a symbolic system, then the substituted system is generally not conjugate to the original system. The argument of [24, 4.2] goes as follows: given $(S, \sigma|_S)$ of minimal block growth either $(S, \sigma|_S)$ is totally minimal (in which case we have a subshift conjugate to a Sturmian via a sliding block code), or there is a prime p_0 such that $(S, (\sigma|_S)^{p_0})$ is not minimal. We then need only treat the second case. One can then show that S breaks into p_0 clopen subsets $\{S_0, \ldots, S_{p_0-1}\}$ which are invariant under $(\sigma|_S)^{p_0}$, and in this case $(S, \sigma|_S)$ is conjugate to

$$A : S_0 \times \{0, \ldots, p_0 - 1\} \to S_0 \times \{0, \ldots, p_0 - 1\}$$

$$(s, i) \mapsto \begin{cases} (s, i + 1), & \text{if } i < p_0 - 1, \\ \big((\sigma|_S)^{p_0}(s), 0\big) & \text{if } i = p_0 - 1. \end{cases}$$

Figuratively, this is a finite adding machine structure superimposed on $(\sigma|_S)^{p_0}$. What is important to the proof of [24, 4.2] is that one can realize the first return map to one of these p_0 clopen sets S_i (which is conjugate to $(\sigma|_S)^{p_0}$) by applying a simple substitution to the original $(S, \sigma|_S)$, which at the same time reduces the K as in the definition of minimal block growth, i.e., the block growth decreases in complexity after this substitution. One then repeats the argument for $(S_i, (\sigma|_S)^{p_0})$ until finally one must reach a totally minimal system, possibly a Sturmian (which are unique among all aperiodic minimal subshifts in having $K = 1$). That is to say that after a finite number of stages and corresponding primes $p_0, \ldots, p_k$ one obtains that the original system $(S, \sigma|_S)$ decomposes into $p_0 \cdots p_k = N$ clopen subsets $C_0, \ldots, C_{N-1}$, each of which is invariant under $(\sigma|_S)^N$, which is also the first return map to each C_i. This first return map $(\sigma|_S)^N$ is then conjugate to a Sturmian subshift and hence is conjugate to the return map to a clopen subset of the minimal set of a Denjoy homeomorphism. The suspended flows are therefore topologically equivalent by theorems of Aarts and Martens [1,2], meaning that there is a homeomorphism of the two sets which preserves the orientation of orbits. Hence, the suspension of any aperiodic subshift of minimal block growth is topologically equivalent to a flow on a Denjoy continuum. □

It is well known that the collection of periodic orbits of (Ω, σ) is dense in Ω. Since the orbits of the suspension of Ω_α are not Lyapunov stable (or equicontinuous), it is not possible for periodic orbits to follow these orbits arbitrarily closely. Hence, we should not really think of the periodic orbits as modeling all the orbits of the flow. The following extends some results in [6], where it is shown that the union of all Denjoy minimal sets in the full shift is dense in the full shift on two symbols. (In [6] a Denjoy minimal set is allowed to have more than one pair of asymptotic orbits, but here we have just one such pair.)

Theorem 2. *For any given $\alpha \in [0,1] - \mathbf{Q}$, the collection of points of Σ belonging to a minimal set which in the suspension flow is homeomorphic to the Denjoy set D_α is a dense subset of Σ.*

Proof. Let $\mathbf{x} = \langle x_n \rangle_{n \in \mathbf{Z}} \in \Sigma$ and let $w = [x_{-N}, \ldots, x_N]$ be a central word of $\mathbf{x}$. Let $\mathcal{A}$ be the alphabet of words of length $2N + 1$ from Σ and let $(\Omega_{\mathcal{A}}, \sigma_{\mathcal{A}})$ be the full shift on the bisequences of $\mathcal{A}$. Consider then the $(2N + 1)$ higher power code $\gamma : \Sigma \to \Omega_{\mathcal{A}}$ given by

$$\langle z_n \rangle_{n \in \mathbf{Z}} \;\xrightarrow{\;\gamma\;}\; \cdots \begin{bmatrix} z_{-(N+1)} \\ \vdots \\ z_{-(3N+1)} \end{bmatrix} \cdot \begin{bmatrix} z_{-N} \\ \vdots \\ z_N \end{bmatrix} \begin{bmatrix} z_{N+1} \\ \vdots \\ z_{3N+1} \end{bmatrix} \cdots.$$

It then follows that $\gamma \circ \sigma^{(2N+1)} = \sigma_{\mathcal{A}} \circ \gamma$ (see, e.g., [21, Section 1.4]). Then form a Sturmian sequence $\mathbf{y}$ from Ω_α on the symbol corresponding to w and some other symbol in $\mathcal{A}$. Then the closure of $\mathbf{y}$ in $\Omega_{\mathcal{A}}$ will be a $\sigma_{\mathcal{A}}$ minimal set M conjugate to the shift on

Ω_α. Then $M' = \gamma^{-1}(M)$ will have a point (bisequence) agreeing with $\mathbf{x}$ on the central block corresponding to w. In the suspension flow, this set will have a time $2N + 1$ map on a Cantor set cross-section that is conjugate to the shift on Ω_α as follows from the relation $\gamma \circ \sigma^{(2N+1)} = \sigma_{\mathcal{A}} \circ \gamma$. And so rescaling time by this factor in the suspension yields a homeomorphism between the suspended minimal set and the suspension of Ω_α, namely D_α. We have essentially realized a sequence from a substituted image of Ω_α agreeing with the original sequence on a central N block. By choosing N large enough we may thus find points from a minimal set homeomorphic to D_α arbitrarily close to $\mathbf{x}$. $\square$

5. Templates

A *template* is a compact branched 2-manifold with boundary together with a smooth expansive semi-flow. The example we study here is called the *Lorenz template* and is shown in Fig. 1. The semi-flow proceeds downward from the branch line the splits and loops around. The orbits merge at the branch line and many orbits exit just below the middle portion of the branch line. The non-wandering set of the semi-flow is locally an interval cross a Cantor set, except for points in the branch line where it is homeomorphic to the product of a "Y" and a Cantor set. We shall take the intersection of the branch line and the non-wandering set to be the middle thirds Cantor set, associating the sequence $\langle x_0, x_1, \ldots\rangle$ with $\sum_{i=0}^{\infty} \frac{2x_i}{3^{i+1}}$. The first return map for this invariant Cantor set is the one-sided (right) shift on 2 symbols:

$$\langle x_0, x_1, \ldots\rangle \mapsto \langle x_1, x_2, \ldots\rangle.$$

There is an extensive literature on *template theory*. Templates where introduced to study *strange attractors* by Williams [28] but are used to model other types of invariant sets in flows. The template form used here is a model for a chaotic saddle set in a *Smale flow*; see, e.g., [27]. Etnyre and Ghrist [9] have used templates to model flows induced by *contact structures* with an eye towards applications in hydrodynamics, while Gilmore and others have used templates to study attractors in various time series data; see [15].

Templates are constructed from invariant sets of hyperbolic flows on 3-manifolds as follows. An isolating neighborhood is foliated by strong stable manifolds of orbits.

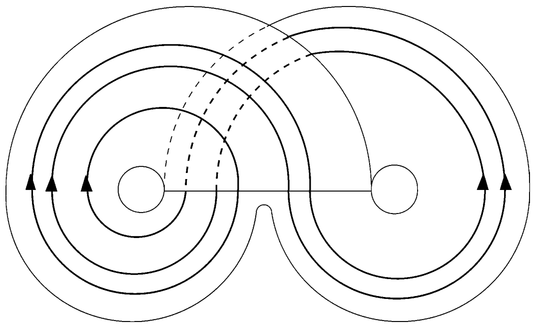

Fig. 1. Lorenz template.

Collapsing along the stable direction results in a branched 2-manifold with an induced semi-flow. The original invariant set can be recovered by an inverse limit. In the collapsing many orbits are identified. But periodic orbits, and the manner in which they are knotted and linked, are preserved in the template model. This was proven in [5], and a proof can also be found in the book [14, p. 38]. In the latter reference it is noted that on the level of the symbolic dynamics the collapsing identifies those orbits approach each other asymptotically in forward time [14, p. 42]: two orbits $\langle x_i \rangle_{i\in\mathbf{Z}}$ and $\langle y_i \rangle_{i\in\mathbf{Z}}$ of the invariant set collapsed onto the branch line are identified on the branch line if and only if $x_i = y_i$ for $i \geqslant 0$. It is important to note that the collapsing takes place along the stable manifolds, and thus the collapsed template can be obtained by homotoping the original invariant set into its collapsed form. Also, since any two distinct minimal sets do not have asymptotic orbits in common, the collapsing only identifies orbits within individual minimal sets. Thus, the "Fundamental Theorem of Templates" extends to other, aperiodic, minimal sets.

Theorem 3. *Given a flow ϕ on a 3-manifold M with a hyperbolic chain-recurrent set, the collection of minimal sets is in bijective correspondence with the collection of minimal sets of the corresponding template*(*s*). *And, for any pair of minimal sets of minimal block growth in the same component of the chain recurrent set of ϕ, the Smith normal form of the linking homomorphism is the same as the Smith normal form of the linking homomorphism of the corresponding minimal sets in the template model.*

All previous work on template theory has focused on the study of the periodic orbits. This is the first paper to examine aperiodic minimal sets.

6. Sturmian links in the Lorenz template

Now we apply the theory to calculate linking matrices for minimal sets of minimal block growth in the Lorenz template, focusing on Sturmian minimal sets. First we develop a convenient way of describing tubular neighborhoods in the template. It is to be recalled that each Ω_α has two asymptotic orbits corresponding to $\mathbf{0}^+$ and $\mathbf{0}^-$ whose forward orbits are eventually identified in the template. However, by Theorem 3, the linking for the collapsed minimal sets in the template S_α and the original are the same. Hereafter, we shall only consider the collapsed minimal sets as they occur within the template.

Definition 1. For a given word $w = w_0 \dots w_{n-1}$, let $[w]$ be the cylindrical w-neighborhood given by the smallest closed segment of the branch line containing all words starting with w together with the forward orbit of all such points up to and including the first return to the branch line.

Definition 2. Given a minimal set or word X in a shift, $\mathcal{L}_n(X)$ denotes the collection of words of length n occurring in X.

This allows us to define the following sequence of neighborhoods of the suspended Sturmian minimal set S_α in the template.

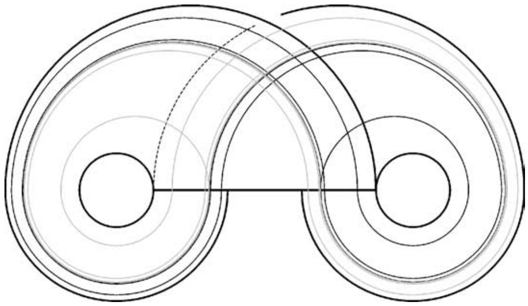

Fig. 2. Two Sturmian minimal sets.

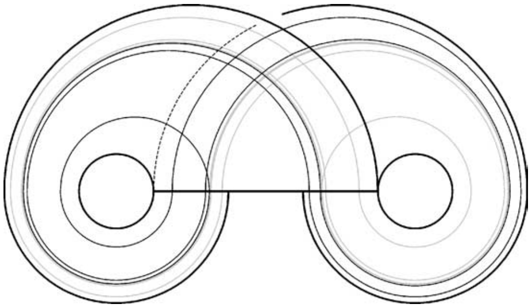

Fig. 3. Trefoil orbit and $S_{3^{-2/3}}$.

Definition 3. Given an $n \in \{1, 2, \ldots\}$, let

$$U_n^\alpha \stackrel{\text{def}}{=} \bigcup \bigl\{[w] \mid w \in \mathcal{L}_n(\Omega_\alpha)\bigr\}.$$

Then $\bigcap_n U_n^\alpha = S_\alpha$. The linking of S_α and S_β can then be measured by finding an n for which U_n^α and U_n^β are disjoint and then measuring the linking of these neighborhoods, which is possible since S_α and S_β are compact and disjoint. The computer plots that follow illustrate the neighborhoods U_5^α for various S_α, which then allows a calculation of the linking.

Example 1. Fig. 2 is an overlay of $S_{12^{-1/3}}$ (black) and $S_{6^{-1/3}}$ (gray). The Smith normal form of the linking of these two minimal sets is $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Example 2. The orbit $\overline{00101}$ is a trefoil knot. Let $\alpha_1 = 3^{-2/3}$. Then from Fig. 3 we obtain a linking matrix $\lambda(\overline{00101}, S_{\alpha_1})$ with Smith normal form $[1, 0]$. In general the linking between a closed orbit and a Sturmian minimal set can be characterized by a single number, the least common divisor of the entries in any linking matrix (vector).

However, a complete justification of this procedure would involve showing that the (co)homology of these cylinder neighborhoods are isomorphic to the (direct) inverse limit of the systems as described in Section 3. In order to develop an inverse limit representation of S_α naturally related to the α and to justify the linking calculations and to describe limitations of the linking of Sturmians, we now we bring to bear facts particular to Sturmian minimal sets. These properties may be found in [10] and are listed here for convenience.

In general, $\mathcal{L}_n(\Omega_\alpha)$ has $n+1$ elements. Thus, exactly one of the n elements of $\mathcal{L}_{n-1}(\Omega_\alpha)$ can be extended with either a 0 or 1 to form words in $\mathcal{L}_n(\Omega_\alpha)$, while all other words are uniquely extended. It is also known (see, e.g., [10, 6.6.19]) that $\mathcal{L}_n(\Omega_\alpha)$ is closed under palindromes, meaning that $w_1 \dots w_n \in \mathcal{L}_n(\Omega_\alpha)$ if and only if $w_n \dots w_1 \in \mathcal{L}_n(\Omega_\alpha)$.

Definition 4. The unique word in $\mathcal{L}_n(\Omega_\alpha)$ that can be extended in two ways on the right to form a word in $\mathcal{L}_{n+1}(\Omega_\alpha)$ is denoted r_n^α, while ℓ_n^α denotes the unique word in $\mathcal{L}_n(\Omega_\alpha)$ that can be extended in two ways on the left.

These ambiguously extended words can be identified with the help of the two asymptotic sequences:

$$\mathbf{0}^+_{n\geqslant 0} = 0^+_0 0^+_1 0^+_2 0^+_3 \dots = 01u_2u_3\dots \quad \text{and}$$
$$\mathbf{0}^-_{n\geqslant 0} = 0^-_0 0^-_1 0^-_2 0^-_3 \dots = 10u_2u_3\dots.$$

Then $iu_2u_3\dots u_n$ for $i \in \{0,1\}$ are both in $\mathcal{L}_n(\Omega_\alpha)$, implying that $u_2u_3\dots u_n = \ell^\alpha_{n-1}$ and that r^α_{n-1} is the palindrome of ℓ^α_{n-1}. In the computer plots, one can detect the *wedge point* $\sum_{i=1}^\infty \frac{2u_{i+1}}{3^i}$, where the two asymptotic orbits merge on the branch line, by the coincidence of the terminus of the two cylinders $[i\ell_4^\alpha]$, $i=1,2$ at the initial part of the cylinder $[\ell_5^\alpha]$. One can also see that exactly one cylinder set has a terminus coinciding with the initial part of two cylinders, seen as a splitting: $[r_5^\alpha]$ feeds into $[r_4^\alpha i]$, for $i=1,2$.

For any given α, $\mathcal{L}_2(\Omega_\alpha)=3$, and an examination of the sequences $\mathbf{0}^+$ and $\mathbf{0}^-$ reveals that $\{01,10\} \subset \mathcal{L}_2(\Omega_\alpha)$. Thus, the following notion is well-defined.

Definition 5. A Sturmian minimal set Ω_α or one of its elements is of *type* 0 or of *type* 1 according as $00 \in \mathcal{L}_2(\Omega_\alpha)$ or $11 \in \mathcal{L}_2(\Omega_\alpha)$, denoted $\tau(\Omega_\alpha)=0$ or 1 accordingly.

Definition 6. For $i=0,1$ let σ_i be the substitution on Ω induced by the function of $\{0,1\}$

$$\sigma_i(i)=i; \qquad \sigma_i(i')=i'i$$

where $i'=1-i$ mod 2. That is,

$$\sigma_i\big(\langle \dots x_{-2}x_{-1}\,.\,x_0x_1\dots\rangle\big) = \big\langle \dots \sigma_i(x_{-2})\sigma_i(x_{-1})\,.\,\sigma_i(x_0)\sigma_i(x_1)\dots\big\rangle.$$

Definition 7. For $u \in \Omega_\alpha$ of type $i \in \{0,1\}$, let $\phi(u)$ be the unique $v \in \Omega$ with either $\sigma_i(v)=u$ or $\sigma(\sigma_i(v))=u$.

Definition 8. The *additive coding sequence* of Ω_α is the sequence

$$\big\langle \tau\big(\phi^n(u)\big)\big\rangle_{n=0}^\infty$$

for any $u \in \Omega_\alpha$.

Definition 9. If the additive coding sequence of Ω_α is written

$$0^{a_0}1^{a_1}0^{a_2}\ldots,$$

where $a_0 \geqslant 0$ and $a_i \geqslant 1$ for $i > 0$ denote the number of consecutive 0's (i even) or 1's (i odd) occurring in the corresponding portion of the additive coding sequence, then the sequence

$$\langle a_i \rangle_{i=0}^{\infty}$$

is the *multiplicative coding sequence.*

Theorem 4. *The multiplicative coding sequence $\langle a_i \rangle_{i=0}^{\infty}$ of Ω_α is equivalent to the continued fraction expansion of α; i.e., the continued fraction expansion of α and $\langle a_i \rangle_{i=0}^{\infty}$ have a common tail* [10, 6.4.23].

We now describe a way of collapsing the cylindrical neighborhoods in such a way as to obtain a natural inverse limit representation of the S_α. Similar constructions provide a systematic way of calculating the linking matrix, yielding very general descriptions of the linking of different types of Sturmians.

By identifying to a point all points within the cylinder [0] that are in the same suspension flow time from the branch line and similarly for [1], we obtain $S^1 \vee S^1 = X_1$ with the branch line (which corresponds to the terminus of both [0] and [1]) mapping to the wedge point and each of the cylinders yielding one of the circles. At the same time this provides a projection of S_α to X_0. We assume now without loss of generality that S_α is of type 0. The neighborhood U_2^α has three cylinders: [00], [10] and [01]. As indicated in Fig. 6, we can find a subtemplate of the original Lorenz template, where this subtemplate has an extra full twist on the right, 1 side. In the terminology of [14], the subtemplate is of type $\mathcal{L}(0, 2)$, just as the subtemplate in [14, 2.4.7].

We now form a wedge of two circles X_1 in much the same way. We identify to a point all points within the cylinder [00] that are in the same suspension flow time from the branch line. Since 11 is not an allowed word for a type 0 Sturmian, there are two cases: (a) $[r_2^\alpha] = [00]$ or (b) $[r_2^\alpha] = [10]$. In case (a), this process identifies the initial segment of the cylinders [00] and [01] to a single point. Then we apply a similar process to the cylinders [01] and [10]. In case (b) we see that the cylinder leading from [10] feeds into the same pair of cylinders [00] and [01]. This identification then leads to a wedge of two circles in either case: one circle corresponding to the cylinder [00] and the other corresponding to [01] and [10]. In either case, the initial segment $[0, \frac{1}{3}]$ along the original branch line corresponds to the wedge point. Also, we can see that the inclusion of the uncollapsed cylinders on this level into the preceding level naturally induces a map $f_1 : X_1 \to X_0$ that can be represented by the matrix $J \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Notice that this map induces an isomorphism of (co)homology and an isomorphism of fundamental groups.

This subtemplate, has a natural symbolic representation where the portion of the original branch line corresponding to $[0, \frac{1}{9}]$ is recoded as 0 and the portion corresponding to $[\frac{2}{9}, \frac{1}{3}]$ is recoded as 1, as indicated in Fig. 6. The symbolic representation of the points of the original Sturmian with respect to this new coding corresponds to the sequences in $\phi(\Omega_\alpha)$

as described in Definition 8. It is important to note that the subshift corresponding to the recoding of the original Sturmian is again Sturmian.

Now we treat this subtemplate and the original minimal set within this subtemplate just as we did the original template. The recoded Sturmian is of type 0 or 1 according as the original Sturmian has additive coding sequence beginning with 00 or 01. See Fig. 7 for a picture of the subsubtemplate corresponding to the 01 case. In either case, after identifying points in the same way as before, we are led to a wedge of two circles X_2 and a map $f_2 : X_2 \to X_1$ represented by the matrix J or its transpose J^{T}, according as we are in the 00 or 01 case.

Repeating this process iteratively, we obtain an inverse sequence (X_i, f_i) with inverse limit $\varprojlim(X_i, f_i)$ homeomorphic to S_α since the cross-sectional diameter of the cylinders goes to 0 as $i \to \infty$, as can be seen by recalling that any cylinder feeds into at most two cylinders. Thus, for any given $\varepsilon > 0$, for sufficiently large i the projection $S_\alpha \to X_i$ is an ε-map. Notice that the bonding maps of both types induce isomorphisms of fundamental groups and (co)homology.

Notice the similarity of this inverse limit representation with that found in [3]. The number of bonding maps in a row of the form J or J^{T} is determined by the multiplicative coding sequence for α and thus is determined by the continued fraction expansion of α by Theorem 4. It follows that if α and β have continued fraction expansions with a common tail, then the corresponding S_α and S_β are homeomorphic. This and its converse are shown in [3,11] for the uncollapsed Denjoy minimal sets D_α and D_β. For the purposes of topologically classifying the suspension of Sturmian minimal sets, only the tail ends of the continued fraction expansion are relevant. However, we shall soon see that only the *beginnings* of the inverse limit expansions are relevant for the linking. To determine the linking of two Sturmians S_0 and S_1 with additive coding sequences $\langle\alpha_i\rangle_{i=0}^{\infty}$ and $\langle\beta_i\rangle_{i=0}^{\infty}$, with $\alpha_i = \beta_i$, $i \leqslant k$, and $\alpha_{k+1} \neq \beta_{k+1}$, we take subtemplates of type α_1, then of type $\alpha_2, \ldots, \alpha_k$. Then the recoded Sturmians will be of different types. In principle, as will become evident below, this then allows us to calculate the linking matrix.

The first proposition foreshadows the style of the arguments to follow and is of independent interest.

Proposition 1. *Let γ be the periodic orbit for $(01)^\infty$. Let S_α be any Sturmian minimal set. Then*

$$\lambda(S_\alpha, \gamma) \approx [\,1 \quad 0\,].$$

Proof. Without loss of generality suppose S_α is of type 0. Recall that we parameterize the branch line of the Lorenz template from left to right as the closed unit interval $[0, 1]$. Let $p \stackrel{\text{def}}{=} \gamma \cap [0, \frac{1}{3}]$, with associated sequence $010101\ldots$ and let $q \stackrel{\text{def}}{=} \gamma \cap [\frac{2}{3}, 1]$, with associated sequence $101010\ldots$. Since any 1 in the sequence for a point of S_α along the branch line is followed by a 0, any point of $S_\alpha \cap [0, \frac{1}{3}]$ is to the left of p and any point of $S_\alpha \cap [\frac{2}{3}, 1]$ is to the left of q. Consider the first subtemplate neighborhood for S_α as described above, with one "tube" corresponding to the cylinder [00] and the other to [01] together with [10]. The portion of S_α in the [00] tube and its first return to the branch line is entirely to the left of γ. Thus, it does not link at all with γ, and so $\lambda(\gamma, S) \approx [n, 0]$, with

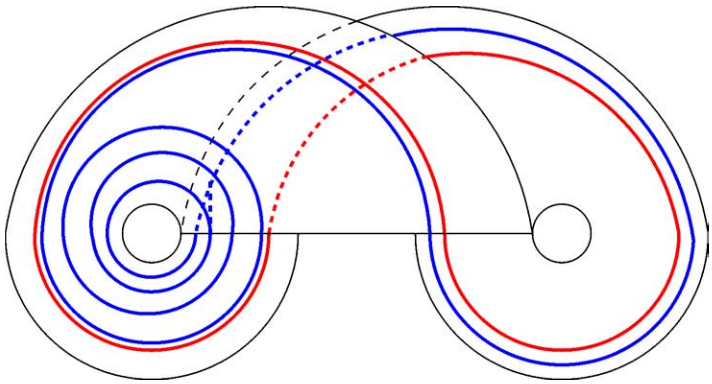

Fig. 4. A Sturmian and the $(01)^{\infty}$ orbit.

the n corresponding to the linking of the portion of S_α in [01], [10] tube. Since γ and S_α satisfy the branch line ordering described above, we can choose the tubular neighborhood corresponding to [01], [10] to be entirely to the left (at the branch line) of γ. This tube then has only one over-crossing with γ. Thus, $\lambda(\gamma, S) \approx [1, 0]$. See Fig. 4 for a typical example. □

There seems to be no such rigidity in the linking of other minimal sets and γ or between Sturmian minimal sets and other periodic orbits. The periodic orbit of $(001)^{\infty}$ and $S_{1/\sqrt{3}}$ have linking matrix with normal form $[1 \quad 0]$, while the same periodic orbit and $S_{\sqrt{2}/3}$ have linking matrix with normal form $[2 \quad 0]$.

The next few propositions and examples explore the 2×2 linking matrix of pairs of Sturmian minimal sets. Computer plots are helpful, but the images quickly become impossible to resolve visually when two Sturmians share the first few terms in their additive sequence. (An illustration of why we should not let our students become too dependent on graphing calculators.)

Proposition 2. *Let S_0 and S_1 be Sturmian minimal sets of type* 0 *and* 1*, respectively. Then*

$$\lambda(S_0, S_1) \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. Since points of $S_1 \cap [0, 1]$ have no consecutive 0's, as in Proposition 1 the left most point of $S_1 \cap [0, \frac{1}{3}]$ is to the right of the right most point of $S_0 \cap [0, \frac{1}{3}]$ and the left most point of $S_1 \cap [\frac{2}{3}, 1]$ is to the right of the right most point of $S_0 \cap [\frac{2}{3}, 1]$. A typical example is illustrated in Fig. 5. Then we can measure the linking of S_0 and S_1 by examining the neighborhood corresponding to the [00] tube and [01], [10] tube of S_0 and the [11] tube and [10], [01] tube of S_1 since all portions of the minimal sets within the [01], [10] tubes are on opposite cross-sectional ends. Then the [00] tube of S_0 does not link at all with S_1.

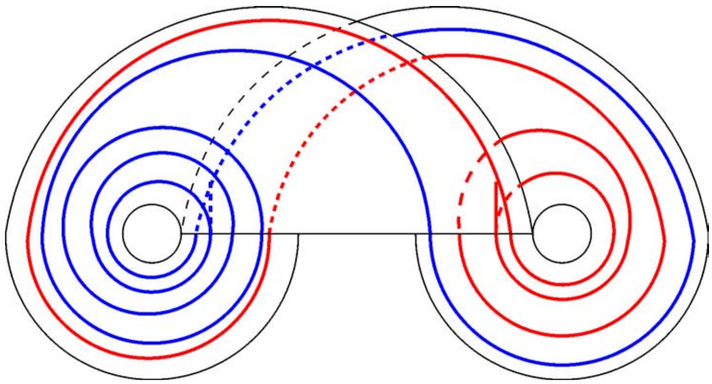

Fig. 5. Sturmians of type 0 and 1.

Similarly, the [11] tube of S_1 does not link with S_0. The [01], [10] tube of S_0 crosses over the [10], [01] tube of S_1 once, and so

$$\lambda(S_1, S_2) \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

Each of the following propositions naturally yields another proposition obtained by reversing the roles of 0 and 1.

Corollary 1. *If we replace the Lorenz template $\mathcal{L}(0,0)$ in Proposition 2 with $\mathcal{L}(0,2n)$, $n > 0$, then the same conclusion holds.*

Proof. With an even number of twists the lexicographical ordering of the branch line and returns to it works as before. Now the long tube of S_0 crosses over the [11] tube of S_1 n times, and it crosses over the long tube of S_1 $n+1$ times. The [00] tube of S_0 still misses S_1. Thus,

$$\lambda(S_0, S_1) \approx \begin{bmatrix} n & 0 \\ n+1 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

Proposition 3. *For $i = 0, 1$ let S_i be Sturmian minimal sets with additive coding sequences beginning with m_i consecutive 0's satisfying $m_0 > m_1 \geqslant 0$. That is, the additive coding sequences are of the form $0^{m_i}1\ldots$. Then*

$$\lambda(S_0, S_1) \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. As in the formation of the inverse sequence, we iteratively form m_1 type 0 subtemplates of the original template. This is then a template of type $\mathcal{L}(0, 2m_1)$. On this subtemplate, the recoded Sturmian systems for S_0 and S_1 are of type 0 and type 1, respectively. Thus, by Corollary 1 we obtain the desired result. $\square$

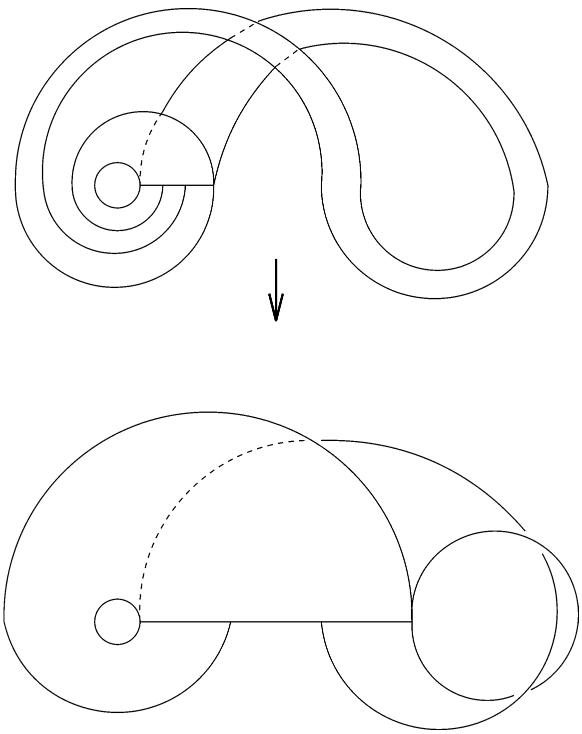

Fig. 6. $L(0,2)$.

Proposition 4. *Let S_0 and S_1 be Sturmian minimal sets whose additive recoding sequences start with* 010 *and* 011*, respectively. Then*

$$\lambda(S_0, S_1) \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. It is now difficult to visualize S_0 and S_1 distinctly on the Lorenz template. As indicated in Figs. 6 and 7, we first take a type 0 and then a type 1 subtemplate. On this subsubtemplate, S_0 is of type 0 and S_1 is of type 1. Then Fig. 8 shows a choice of tubes systems, where the [00] and [01] cylinders are conflated below the branch line for the ease of computer drawing. (This has no effect on the linking calculation.) In this and the following figure, a small box with the number N in it represents N half-twists of the band inside the box. This yields:

$$\lambda(S_0, S_1) \approx \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

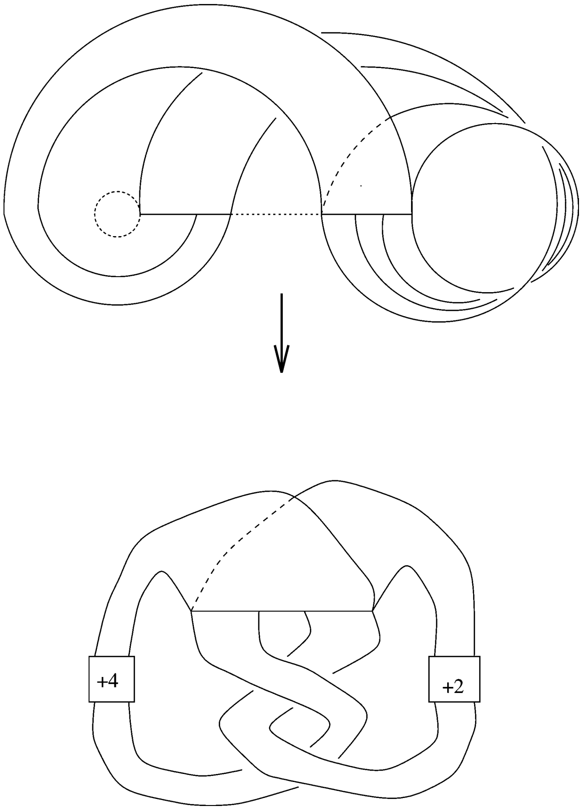

Fig. 7.

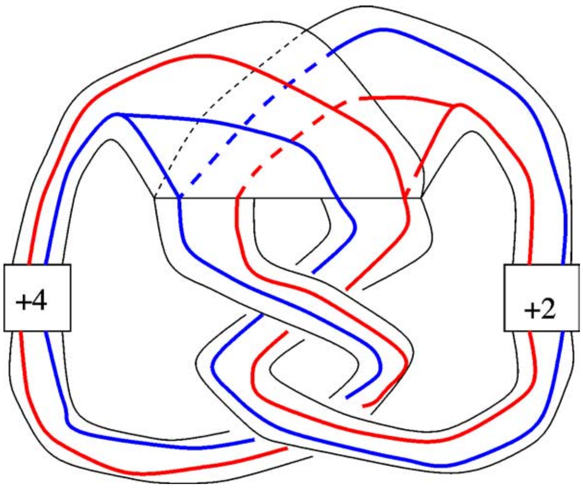

Fig. 8. Tubular neighborhoods.

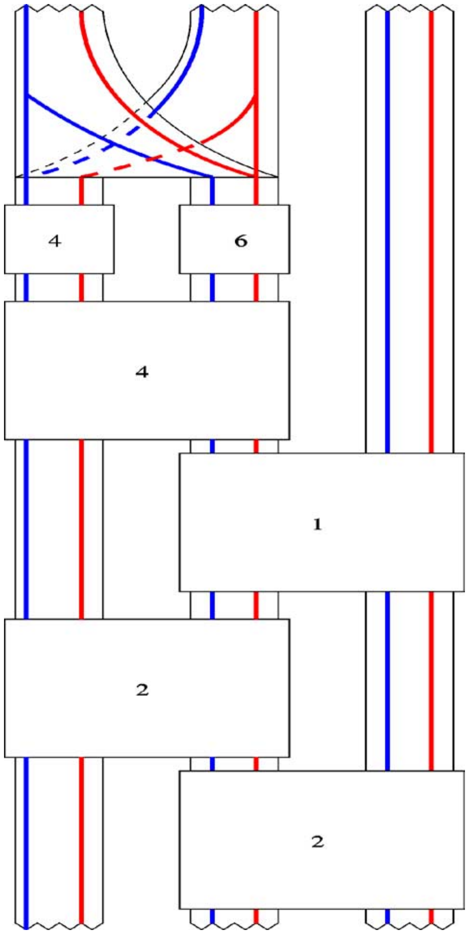

Fig. 9. Template for Proposition 5.

Proposition 5. *Let S_1 and S_2 be Sturmian minimal sets whose additive recoding sequences start with* 0100 *and* 0101, *respectively. Then*

$$\lambda(S_1, S_2) \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. A further iteration of the procedure used in Proposition 4 yields the subsubsubtemplate and tube systems shown in Fig. 9. (In this figure and the next, a box covering two bands with a number M represents M band crossings, left over right, with no twisting.) Thus,

$$\lambda(S_1, S_2) \approx \begin{bmatrix} 3 & 5 \\ 9 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

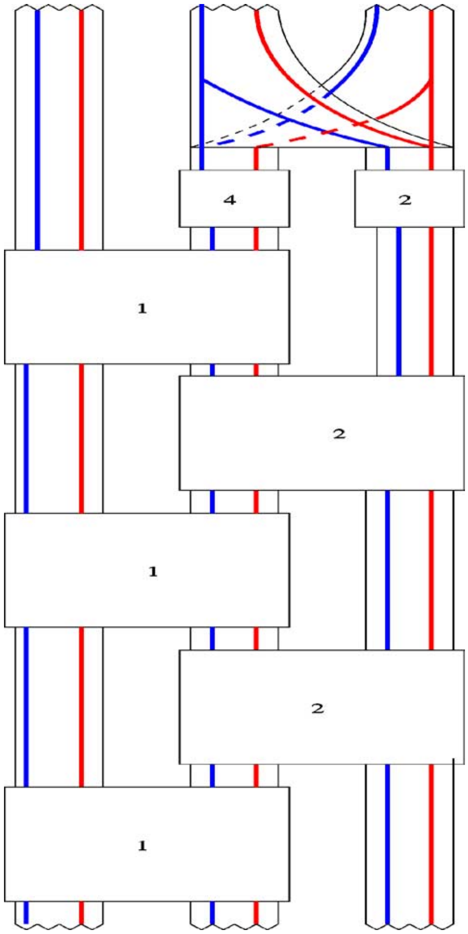

Fig. 10. Template for Proposition 6.

The reader may be wondering if the linking matrix for any pair of Sturmian minimal sets is $\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Indeed, for a time we had hoped to prove that this was the case. However, the following shows this is not so.

Proposition 6. *Let S_0 and S_1 be Sturmian minimal sets whose additive recoding sequences start with* 0110 *and* 0111*, respectively. Then*

$$\lambda(S_1, S_2) \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. A further iteration of the procedure used in Proposition 4 yields the template and tube systems shown in Fig. 10. Thus,

$$\lambda(S_1, S_2) \approx \begin{bmatrix} 2 & 7 \\ 3 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad \square$$

Non-Sturmian minimal sets of minimal block growth are much more flexible in their linking behavior. For example, if one applies the substitution $\theta : 0 \overset{\theta}{\mapsto} 0100;\ 1 \overset{\theta}{\mapsto} 0011$ to the Fibonacci substitution minimal set, one obtains a minimal set M of minimal block growth. The linking matrix of the suspension of M and $S_{1/\sqrt{2}}$ has Smith normal form

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The apparent simplicity of the Smith normal forms for linking matrices of pairs of Sturmian minimal sets in the Lorenz template is surprising and intriguing.

Question. Given the additive coding sequence of two Sturmian minimal sets, what is the Smith normal form of the matrix representing their linking?

While we do not currently have an answer to this question, our procedure for taking subtemplates of the appropriate type until one reaches a subtemplate for which the two minimal sets are of different types does lead to the following general observation.

Theorem 5. *Any two Sturmian minimal sets in the Lorenz template have a linking matrix with non-zero Smith normal form and so are essentially linked.*

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