LINEAR VECTOR FIELDS

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It happens on occasion that a student in my multivariable calculus class (Calculus III at my university) has already had linear algebra. And, sometimes it even transpires that a student in my sophomore linear algebra class has had multivariable calculus. When either of these situations arise, I pull said student aside a talk to them about the divergence and curl of linear vectors fields.

I'd love to find a way to cover the connection between linear algebra and vector fields in either course, but to date I have failed. The textbooks and syllabi are disjoint. The situation reminds me of those plates children use so that different types of food will not touch. Here we outline some talking points that instructors can explore with students who have drunk from both cups and conclude with some open ended questions.

Let A be a 3×3 matrix of real constants and v be the column vector of variables $[x, y, z]^T$. Then $v' = Av$ is a system of linear differential equations. But Av can also be viewed as a vector field and one can apply the usual tools from vector calculus. The following facts are interesting and easy to verify.

• $\mathbf{div} Av$ is the sum of the eigenvalues of A. It is easy to check that $\mathbf{div} Av$ equals the trace of A. Then we recall that the trace of a matrix is invariant under similarity and hence under diagonalization. What's interesting is that this helps students connect the geometric idea behind eigenvalues and the physical idea behind divergence, hopefully re-enforcing both concepts.

• Regarding the set a linear vector fields on $\mathbb{R}^3$ as a nine dimensional vector space under matrix addition, the set of divergence zero fields is an eight dimensional subspace.

• $\mathbf{curl} Av = \mathbf{0}$ if and only if $A = A^T$, thus, if and only if the eigenvalues are real and the eigenspaces are orthogonal. Hence the irrotational fields form a six dimensional subspace. The intersection of the sets of curl and divergence free linear fields is five dimensional.

• If $\mathbf{curl} Av = \mathbf{0}$ then the potential function is

$$\frac{a}{2}x^2 + \frac{e}{2}y^2 + \frac{i}{2}z^2 + bxy + cxz + fyz$$

up to any constant, where $A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}$. This is easy to check. It generalizes that $\int ax\,dx = \frac{a}{2}x^2 + C$ in single variable calculus, but in 3-space we need that assumption that the field is irrotational which can be thought of as being true by default on the real line.

• Now think about our last two observations. Your student should be familiar with the quadratic surfaces. The level sets of the potential functions above are just rotations of these. (Some linear algebra texts cover this fact under the rubric of quadratic forms.) In hindsight, doesn't it make sense that the symmetry of the level surfaces would require the eigenspaces to be real and orthogonal thus forcing $A = A^T$?

• Divergence and curl should behave naturally when we rotate a linear vector field. Let R be a 3×3 rotation matrix. Then from our geometric intuition we expect

$$\mathbf{div} R^{-1} A R v = \mathbf{div} A v$$

and

$$\mathbf{curl} R^{-1} A R v = R^{-1} \mathbf{curl} A v.$$

The first equation holds since A is similar to $R^{-1}AR$ and so they have the same trace. To prove the second you can grind out both sides and then simplify using two facts about rotation matrices, $RR^T = I$ and $\det R = 1$. The calculations are messy but straight forward.

• It is shown in most vector calculus texts that for a pure rotation about the z-axis the curl is normal to the xy-plane with magnitude twice the angular velocity. We can generalize this to the case were we have complex eigenvalues $\lambda \pm i\mu$ and the real eigenvector is normal to the invariant plane spanned by the real and imaginary parts of either of the complex eigenvectors. The real eigenvalue, and the real parts of the complex eigenvalues have no effect on curl; it will be normal to the invariant plane (use our rotations result) and have magnitude $2|\mu|$; its direction still follows the right hand rule.

But as soon as the real eigenspace is no longer normal to the invariant plane, the direction and magnitude of the curl vector cease to be obvious. I leave you with this challenge.

In the case where the matrix has a pair of complex eigenvalues find a relationship between the curl vector and the direction of the real eigenspace when it is not perpendicular to the invariant plane.

For example you might start with

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix rotates vectors in the xy-plane by θ radians and fixes the z-axis, that is the z-axis is an eigenspace with eigenvalue one. Then $\mathbf{curl}Av = (2\sin\theta)\mathbf{k}$. What happens to the curl when we pull the real eigenspace a little toward the x-axis, say with azimuthal angle ϕ? Can you develope an intuitive sense for the behavior of curl? First you'll need to figure how A changes.

If the eigenvalues are real it would also be interesting to discover an intuitive understanding of the curl. One might start with some irrotational linear vector fields and then observe where the curl vector appears when small changes are made.

I wrote a Maple program that for a given 3×3 matrix will plot real eigenvectors (or generalized eigenvectors) and an invariant plane when there is a pair of complex eigenvalues, and the curl vector. A typical output is shown in Figure 1, where input matrix was $\begin{bmatrix} -2 & 2 & -3 \\ 2 & -2 & 1 \\ 3 & -2 & -1 \end{bmatrix}$. The program also reports that the complex eigenvalues are $-2.159891038 \pm 2.171695094 * I$; that the invariant plane is given by $-.2459118507e - 1 * x + .8229181570 * y - .5676275016 * z = 0$; that the real eigenvalue is 0.3197820749 with eiganspace spanned by $[-.2376202145e - 1 + 0. * I, -.331739142e - 1 + 0. * I, -.3741705e - 2 + 0. * I]$; and finally that the curl vector is $[-1., -6., 0.]$. In Figure 1 the curl vector is shown based at the origin. Of course the curl field is this vector at each point of $\mathbb{R}^3$.

I played with many other examples but never did find a pattern I could state as a theorem. Maybe you or your students will have better luck. You can download the program I wrote from

`http://galileo.math.siu.edu/~msulliva/Curl`

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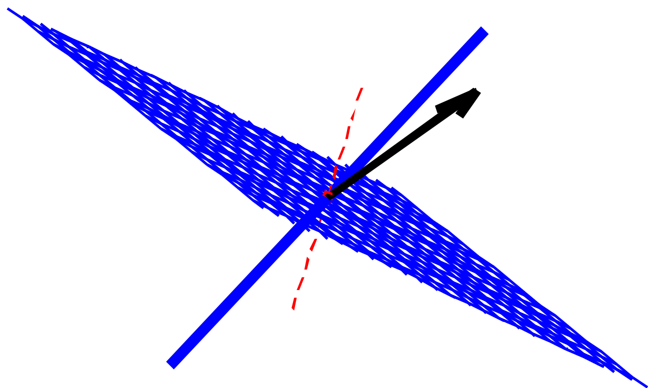

FIGURE 1. An invariant plane with a normal axis (blue), the real eigenspace (dashed red line) and the curl vector (black).