Frenet-Serret formulas and Torsion

We shall work through Problems 43, 45 and 46 in Section 10.8. These lead us to define the torsion of a space curve. At the end we discuss how torsion is a natural extension of the notions of velocity and curvature.

Recall the formulas in the box below.

\[
\begin{align*}
\mathbf{B} &= \mathbf{T} \times \mathbf{N} \\
\mathbf{B} \times \mathbf{T} &= \mathbf{N} \\
\mathbf{N} \times \mathbf{B} &= \mathbf{T} \\
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \\
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \\
(\mathbf{u} \times \mathbf{v})' &= \mathbf{v}' \times \mathbf{u} + \mathbf{v} \times \mathbf{u}'
\end{align*}
\]

43. Show that \( \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \).

*Proof.* By Definition 8 \( |\frac{d\mathbf{T}}{ds}| = \kappa \) so \( \frac{d\mathbf{T}}{ds} \) will have the same magnitude as \( \kappa \mathbf{N} \). Recall we defined \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|} \), but the parametrization used only affects the \( \mathbf{N} \) by a scalar factor of \( \pm 1 \). If arc length \( s \) is defined to be increasing with \( t \) then the sign choice is the same. Thus \( \mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\left|\mathbf{T}'(s)\right|} \). Hence \( \mathbf{N}(s) \) has the same direction as \( \mathbf{T}'(s) = \frac{d\mathbf{T}}{ds} \) and by definition \( \kappa \geq 0 \). Thus, \( \frac{d\mathbf{T}}{ds} \) and \( \kappa \mathbf{N} \) have the same direction and magnitude, so they are equal. \( \square \)

45.a. Show that \( \frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0 \).

*Proof.* Since \( |\mathbf{B}| = 1 \) is a constant, Example 12 from 10.7 gives us that \( \mathbf{B}' \cdot \mathbf{B} = 0 \). \( \square \)

b. Show that \( \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0 \).

*Proof.*

\[
\begin{align*}
\mathbf{B}' \cdot \mathbf{T} &= (\mathbf{T} \times \mathbf{N})' \cdot \mathbf{T} \\
&= (\mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}') \cdot \mathbf{T} \\
&= (\mathbf{T}' \times \mathbf{N}) \cdot \mathbf{T} + (\mathbf{T} \times \mathbf{N}') \cdot \mathbf{T} \\
&= (\kappa \mathbf{N} \times \mathbf{N}) \cdot \mathbf{T} - (\mathbf{N}' \times \mathbf{T}) \cdot \mathbf{T} \\
&= \kappa \mathbf{0} \cdot \mathbf{T} - \mathbf{N}' \cdot (\mathbf{T} \times \mathbf{T}) \\
&= 0 - \mathbf{N}' \cdot 0 \\
&= 0
\end{align*}
\]

\( \square \)

c. Since \( \frac{d\mathbf{B}}{ds} \) is perpendicular to both \( \mathbf{B} \) and \( \mathbf{T} \), it must be parallel to \( \mathbf{N} \). We define \( -\tau(s) \) to be the scaling factor such that \( \frac{d\mathbf{B}}{ds} = -\tau(s) \mathbf{N} \).
d. If a curve lives in a fixed plane then $B$ is always the same unit vector perpendicular to that plane. Since $B$ does not change $\frac{dB}{ds} = 0$. Thus $\tau(s) = 0$ for such a curve. The reverse is true as well. More generally we can think of $\tau(s)$ to be a measure of the tendency of a curve to move away from its osculating plane.

46. Show that $\frac{dT}{ds} = \kappa N$.

Proof. From 43 we have $\frac{dT}{ds} = \kappa N$ and from 45c we know $\frac{dB}{ds} = -\tau N$.

\[
N' = (B \times T)' = B' \times T + B \times T' = (-\tau N) \times T + B \times (\kappa N) = \tau (T \times N) - \kappa (N \times B) = \tau B - \kappa T
\]

\[\square\]

Summary. Let $r(t)$ be a smooth space curve. The three equations in the box below are called the Frenet-Serret formulas.

\[
\begin{align*}
\frac{dT}{ds} &= \kappa N \\
\frac{dN}{ds} &= -\kappa T + \tau B \\
\frac{dB}{ds} &= -\tau N
\end{align*}
\]

Sometimes they are expressed in matrix form.

\[
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}' =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\]

- At any given time the curve has a location. The velocity $v$ is the tendency to move away from the present location. If $v$ is always 0 then we will stay at our present location forever.
- At any given time the curve has a tangent line. The curvature $\kappa$ is the tendency to move away from the present tangent line. If $\kappa$ is always 0 then we will stay on the present tangent line forever.
- At any given time the curve has an osculating plane. The torsion $\tau$ is the tendency to move away from the present osculating plane. If $\tau$ is always 0 then we will stay on the present osculating plane forever.

However, velocity depends on the parametrization whereas curvature and torsion are geometric properties of the curve independent of the parametrization.