Properties of the Real Number System

• Algebraic Properties.
  \( \mathbb{R} \) is an ordered field, with \( \mathbb{Q} \) an ordered subfield, and \( \mathbb{Z} \) an ordered commnunative subring with a unit.

• Cardinality.
  \( \mathbb{R} \) is uncountably infinite, while \( \mathbb{Q} \) and \( \mathbb{Z} \) are countably infinite.

• Dense Subsets.
  Between any pair of distinct real numbers there is a rational number and an irrational number.

• Archimedean Properties.
  For every \( x \in \mathbb{R} \) there exists an \( n \in \mathbb{N} \) such that \( n > x \).
  For every \( x \in (0, \infty) \) there exists an \( n \in \mathbb{N} \) such that \( \frac{1}{n} < x \).

• \( \epsilon \)-Principle.
  If \( \forall \epsilon > 0 \) we have \( a \leq b + \epsilon \), then \( a \leq b \).
  If \( \forall \epsilon > 0 \) we have \( |x - y| \leq \epsilon \), then \( x = y \).

• Completeness Properties.
  If \( S \subset \mathbb{R} \) has an upper bound, then \( S \) has a least upper bound; it is called the supremum of \( S \) and is denoted \( \sup S \). If \( S \neq \emptyset \) and has no upper bound then we define \( \sup S = \infty \). We define \( \sup \emptyset = -\infty \).
  If \( S \subset \mathbb{R} \) has a lower bound, then \( S \) has a greatest lower bound; it is called the infimum of \( S \) and is denoted \( \inf S \). If \( S \neq \emptyset \) and has no lower bound then we define \( \inf S = -\infty \). We define \( \inf \emptyset = \infty \).

• Subsets.
  If \( \emptyset \neq S \subset T \), then
  \[ -\infty \leq \inf T \leq \inf S \leq \sup S \leq \sup T \leq \infty. \]

• Cauchy Completeness.
  An infinite sequence of real numbers \( (a_n) \) converges iff
  \[ \forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies |a_n - a_m| < \epsilon. \]

• Existence of Roots.
  For every \( x \in [0, \infty) \) and \( n \in \mathbb{N} \), there exists a unique \( y \in [0, \infty) \) such that \( y^n = x \).

• Triangle Inequality.
  In \( \mathbb{R}^n \) we have \( |x - z| \leq |x - y| + |y - z| \).